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REPEATABILITY IN REDUNDANT MANIPULATOR SYSTEMS

by

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REPEATABILITY IN REDUNDANT MANIPULATOR SYSTEMS[*]

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Abstract

Terrestrial manipulators with more dof than the dimension of the workspace and space manipulators with as many manipulator dof as the dimension of the workspace are both redundant systems. An interesting problem of such redundant systems has been the repeatability problem due to the presence of nonholonomic constraints. We show in this paper, contrary to the existing belief, that integrability of the nonholonomic constraints is not a necessary condition for the repeatability of the configuration variables. There exist certain trajectories in the independent configuration variable space that are like "holonomic loops" along which the redundant manipulators exhibit repeatable motion. In this paper we present a simple method based on optimization techniques for designing repeatable trajectories for free-flying space manipulators and terrestrial redundant manipulators under pseudoinverse control.

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1. Introduction

An important problem of kinematically redundant robot manipulators has been the repeatability problem under pseudoinverse control. This problem was initially observed and analyzed by Klein and Huang [7] where resolved motion rate control [22] using the Jacobian pseudoinverse was noticed to result in nonrepeatable joint motion. As an alternative to the pseudoinverse control, Baillieul [2] proposed the extended Jacobian technique that lifted closed end-effector paths to closed joint paths. Later, Seraji [15] proposed the configuration control technique that also resulted in repeatable joint motion. The key idea behind the extended Jacobian technique was to add enough independent additional constraints to the motion of the manipulator so that the rectangular Jacobian of the redundant manipulator could be converted into a square Jacobian of full rank. When the additional constraints that are imposed upon the motion of the redundant manipulator are holonomic in nature, the full rank square Jacobian guarantees repeatability in the joint motion.

Though the pseudoinverse control does not produce repeatable joint motion in general, it is fundamentally similar to the extended Jacobian technique in the sense that it is also a form of constrained motion. For a redundant manipulator with n degrees of freedom and a workspace of dimension m, the dimension of the null space of the manipulator Jacobian is equal to $(n-m)$. The extended Jacobian technique [2] imposes $(n-m)$ additional independent constraints on the motion of the system to make the Jacobian square and full rank. The pseudoinverse control is equivalent to the imposition of $(n-m)$ constraints that direct the motion of the joints orthogonal to the $(n-m)$ dimensional null space. While the constraints due to pseudoinverse control are nonholonomic or nonintegrable, the constraints imposed by the extended Jacobian technique are holonomic or integrable. Therein lies the essential difference between the two approaches, more of which will be discussed in section 2.

The pseudoinverse control problem has been studied by a number of researchers [7], [8], [9], and [16]. Klein and Huang [7] analyzed the nonrepeatability problem of a three link planar redundant manipulator in terms of the integrability condition of a Pfaffian differential form. Shamir and Yomdin [16] asserted that for a redundant manipulator repeatability is guaranteed if and only if there exists an integral surface of the distribution spanned by the column vectors of the Jacobian pseudoinverse. Under the differential geometric framework adopted, it was concluded that the repeatability of a redundant manipulator can be assured if and only if a certain "Lie Bracket Condition" (LBC) is satisfied. In section 3 of this paper we will show that this LBC is not a necessary condition for the repeatability of redundant manipulators. We will also show in section 2 that the LBC is not a sufficient condition for repeatability when applied to arbitrary extended Jacobians. This contradicts some of the discussion by Luo and Ahmad [9] who discussed the measure of repeatability for planar redundant manipulators under pseudoinverse control. They used a framework based on the theory of integration on manifolds. The authors in [7], [9], and [16] have all concluded in essence that integrability is a necessary condition for the repeatability in redundant manipulators. Similar opinion was also expressed in [3]. We do not quite agree with this statement. Our contention is that integrability is only a sufficient condition for repeatability, it is by no means a necessary condition. In section 3 will derive a weaker necessary condition for the repeatability in redundant manipulators.

In 1989 Klein and Kee [8] presented a numerical procedure to find stable drift-free trajectories in redundant manipulators under pseudoinverse control. Later, Klein [6] tried to predict the stable drift-free trajectories of [8] by using the Lie Bracket Condition (LBC) in [16]. The results indicated that the stable trajectories in [8] are not contained in the LBC surfaces of [16]. This bears testimony to the fact that the LBC of [16] is not a necessary condition for repeatability.

Recently Roberts and Maciejewski [14] presented a necessary and sufficient condition for the existence of stable surfaces for repeatable motion in redundant manipulators. They showed that the Lie Bracket Condition (LBC) of [16] is a necessary condition for the existence of an integral surface, but it is not a sufficient condition for the surface to be stable for repeatability. Since stable surfaces are quite rare, the authors [14] designed a repeatable control that is nearest, in an integral norm sense, to a desired optimal control. In this paper we are concerned with repeatable trajectories but not with their stability. Though the LBC is a necessary condition for a stable surface, we show that it is not a necessary condition for repeatability. Using a necessary condition, weaker than the LBC, we will show the existence of "holonomic loops" that lift closed paths in the workspace to closed paths in the joint space under pseudoinverse control.

With no intention of digressing, we would like to mention that space robots with as many manipulator degrees of freedom as the dimension of the workspace exhibit a special kind of redundancy called "nonholonomic redundancy". It was shown that nonholonomic redundancy, unlike ordinary redundancy, manifests itself only after a global motion and is not characterized by "self-motion" manifolds [12]. Inspite of fundamental differences between nonholonomic redundancy and ordinary kinematic redundancy, more of which will be discussed in section 2, the control problem in both is characterized by the non-integrability of the distribution spanned by the vector fields of the system. Hence the repeatability problem in space manipulators with no additional degrees of freedom and terrestrial redundant manipulators are inherently similar.

Since space manipulators are also redundant systems, the repeatability problem in space manipulators fall within the scope of this research. While the problem of reorienting a space multibody system using internal motion has been studied by a number of researchers [10], [11], [13], [18], [21], [23], etc., an important problem that has not been addressed so far is the repeatability problem in space manipulators. The motion of the end-effector of a space manipulator is related to the joint motions through the "generalized Jacobian" [19] by eliminating the dependence of the end-effector motion on the change of orientation of the space vehicle. While the joints of the space robot move along closed paths, the orientation of the space vehicle does not. Consequently the end-effector of the space manipulator does not move along a closed path. And conversely, the joints of the space robot fail to move along a closed path when the end-effector traces a closed path. While a more complete discussion on this topic will follow in section 2.3, we only like to reiterate here that the manifestation of redundancy in terms of nonrepeatability in the configuration variables is observed in nonholonomically redundant space robots and ordinarily redundant terrestrial robots alike.

The rest of the paper is organized as follows. In section 2 we take a look at the control of redundant manipulators, including space manipulators, from a different perspective. In section 3 we derive a necessary condition for repeatability in nonholonomic systems like space manipulators and redundant manipulators under pseudoinverse control. We use this necessary condition to find repeatable trajectories for nonholonomically redundant space manipulators in section 4 and ordinarily redundant terrestrial manipulators in section 5. In section 6 we present some results obtained through computer simulation.

2. A Different Perspective on Redundant Manipulator Control

2.1 Pseudoinverse Control as a form of Constrained Motion

Kinematically redundant manipulators have more degrees of freedom than the dimension of the workspace. For such systems, the direct kinematic relationship can be written in the form

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \tag{1}$$

where $\boldsymbol{x} \in R^m$ represents the workspace variables, $\boldsymbol{\theta} \in R^n$ represents the manipulator's joint variables, and $n > m$ by the definition of redundancy. Differentiating Eq.(1), we get

$$\dot{\boldsymbol{x}} = \boldsymbol{J}\,\dot{\boldsymbol{\theta}}, \qquad \boldsymbol{J} \triangleq \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}}\right) \in R^{m\times n} \tag{2}$$

where $\boldsymbol{J}$ is the manipulator Jacobian matrix. The pseudoinverse solution invokes the control law

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{\#}\,\dot{\boldsymbol{x}} \tag{3}$$

where $\boldsymbol{J}^{\#} \in R^{n\times m}$ is the pseudoinverse of $\boldsymbol{J}$. We will always assume in our discussion that the manipulator is not at any singular configuration. Therefore the Jacobian will always have full rank and the null space of the Jacobian will have a dimension of $(n-m)$. The pseudoinverse solution has the minimum norm property which implies that the joint motion $\dot{\boldsymbol{\theta}}$ obtained from Eq.(3) will have to be orthogonal to the null space of $\boldsymbol{J}$. The orthogonality requirement is a constraint on the joint velocities $\dot{\boldsymbol{\theta}}$. Since the null space of $\boldsymbol{J}$ has a dimension of $(n-m)$, the pseudoinverse solution in Eq.(3) will impose $(n-m)$ velocity constraints. To illustrate this concept we consider the simple three link planar redundant manipulator shown in Fig.1. The lengths of all the links of the manipulator are assumed to be unity for simplicity. The workspace is defined by the Cartesian coordinates X and Y and the manipulator configuration is described by the absolute angles θ_1, θ_2, and θ_3. The direct kinematic relationship, as in Eq.(1), is of the form

$$\begin{aligned} X &= \cos\theta_1 + \cos\theta_2 + \cos\theta_3 \\ Y &= \sin\theta_1 + \sin\theta_2 + \sin\theta_3 \end{aligned} \tag{4}$$

Therefore the Jacobian matrix, as in Eq.(2), is given by

$$J_\theta = \begin{pmatrix} -\sin\,\theta_1 & -\sin\,\theta_2 & -\sin\,\theta_3 \\ \cos\,\theta_1 & \cos\,\theta_2 & \cos\,\theta_3 \end{pmatrix} \tag{5}$$

The Jacobian has a 1-dimensional null space whose basis vector can be conviniently obtained as a cross product of the row vectors of J_θ. The velocity constraint due to the pseudoinverse control can then be expressed as

$$\sin(\theta_3 - \theta_2)\; d\theta_1 + \sin(\theta_1 - \theta_3)\; d\theta_2 + \sin(\theta_2 - \theta_1)\; d\theta_3 = 0 \tag{6}$$

A necessary and sufficient condition for the integrability of a differential expression of the form

$$v_1\, d\alpha + v_2\, d\beta + v_3\, d\gamma = 0 \tag{7}$$

is that [5]

$$v_1\left(\frac{\partial v_2}{\partial\gamma} - \frac{\partial v_3}{\partial\beta}\right) + v_2\left(\frac{\partial v_3}{\partial\alpha} - \frac{\partial v_1}{\partial\gamma}\right) + v_3\left(\frac{\partial v_1}{\partial\beta} - \frac{\partial v_2}{\partial\alpha}\right) = 0 \tag{8}$$

Using the necessary and sufficient condition above, it is quite straightforward to show that the constraint due to pseudoinverse control, given by Eq.(6), is not integrable or nonholonomic in the general case. Therefore the three link manipulator shown in Fig.1 has three expressions of motion under pseudoinverse control: the two kinematic relations given by Eq.(4), and one nonintegrable constraint given by Eq.(6).

In the general case of a redundant manipulator with n joints in an m-dimensional workspace, there are $(n - m)$ nonholonomic constraints (the number of nonholonomic constraints is equal to the degree of redundancy in the system) imposed by pseudoinverse control and m kinematic relations for a total of n expressions of motion. Of course, the nonholonomic nature of the pseudoinverse constraints need to be ascertained from the more general test for integrability in n dimensions, provided in Appendix-A.

When the manipulator has as many degrees of freedom as the dimension of the workspace, *i.e.* $m = n$, the Jacobian is square and has no null space assuming of course that the system is not at any singular configuration. Then the pseudoinverse control does not impose any nonholonomic constraints; the motion of the system is entirely governed by the n direct kinematic relations that are holonomic.

2.2 Repeatability using the Extended Jacobian

Consider a simple nonholonomic system whose constraint equation is of the form

$$dz = a\, dx + b\, dy \tag{9}$$

where x, y, and z are the system variables, and a and b are functions of x and y. The variables x and y can be considered to be the independent variables of the system and z may be considered to be the dependent variable. If x and y move along a closed path, the change in the dependent variable z is expressed as

$$\int dz = \oint_{\partial D} a\,dx + b\,dy = \iint_D \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx dy$$

In the above equation the line integral was conviniently expressed as a surface integral using the generalized Stokes' Theorem [1] on the 2-dimensional oriented manifold D. ∂D is the path of line integration and is the boundary of the domain D. Since the constraint in Eq.(9) is a nonholonomic constraint, we can show

$$\frac{\partial b}{\partial x} \neq \frac{\partial a}{\partial y}$$

by using the test for integrability in Eqs.(7) and (8). Then it simply follows that the dependent variable z does not move along a closed path as the independent variables x and y move along closed paths. Clearly, all the variables of the nonholonomic system in Eq.(9) do not move along closed paths simultaneously. This is true for nonholonomic systems in general including redundant manipulators under pseudoinverse control. The pseudoinverse control of redundant manipulators result in nonrepeatable motion of the joint variables. Conversely, holonomic systems are characterized by repeatability in the configuration variables that can be proven directly from the test for integrability.

The problem of nonrepeatability of nonholonomically constrained redundant manipulators under pseudoinverse control can be remedied by using the extended Jacobian method [2]. For a n joint redundant manipulator with an m-dimensional workspace, the extended Jacobian method imposes $(n-m)$ independent holonomic constraints of the form

$$g(\theta) = 0, \qquad g \in R^{(n-m)} \tag{10}$$

These holonomic constraints are similar to the direct kinematic relations in Eq.(1) and can be used to augment the manipulator Jacobian in Eq.(2) for constructing the extended Jacobian as follows

$$\begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} = J_E \dot{\theta}, \qquad J_E \triangleq \begin{pmatrix} J \\ J_G \end{pmatrix} \in R^{n \times n}, \qquad J_G \triangleq \left(\frac{\partial g}{\partial \theta} \right) \in R^{(n-m) \times n} \tag{11}$$

The above equation indicates that we have artificially increased the dimension of the workspace from m to n and in effect we now have a nonredundant manipulator with a square Jacobian. From our discussion in section 2.1, we know that the pseudoinverse control of such a manipulator does not impose any nonholonomic constraints. The motion of the system is then governed completely by the m direct kinematic relations in Eq.(1) and the $(n-m)$ additional constraints of Eq.(10). Holonomic systems are characterized by repeatability and the extended Jacobian technique achieves repeatability by converting the "redundant-manipulator-pseudoinverse-control" nonholonomic system into a holonomic system.

An alternative way to look at the extended Jacobian technique is to reconsider the constraints in Eq.(11). If we group the joint variables $\boldsymbol{\theta} \in R^n$ into two sets consisting of $\boldsymbol{\theta}_1 \in R^m$ and $\boldsymbol{\theta}_2 \in R^{n-m}$, we can write from Eq.(11)

$$\dot{\boldsymbol{x}} = \boldsymbol{J}_1 \dot{\boldsymbol{\theta}}_1 + \boldsymbol{J}_2 \dot{\boldsymbol{\theta}}_2, \qquad \boldsymbol{J}_1 \triangleq \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}_1}\right), \quad \boldsymbol{J}_2 \triangleq \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}_2}\right) \tag{12}$$

$$0 = \boldsymbol{J}_{G1} \dot{\boldsymbol{\theta}}_1 + \boldsymbol{J}_{G2} \dot{\boldsymbol{\theta}}_2, \qquad \boldsymbol{J}_{G1} \triangleq \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}_1}\right), \boldsymbol{J}_{G2} \triangleq \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}_2}\right) \tag{13}$$

Since the $(n-m)$ constraints in Eq.(10) are all independent, it will be possible to find the set $\boldsymbol{\theta}_2 \in R^{n-m}$ such that the matrix $\boldsymbol{J}_{G2}$ is always invertible. Then Eqs.(13) and (12) can be written as

$$\dot{\boldsymbol{\theta}}_2 = -\boldsymbol{J}_{G2}{}^{-1} \boldsymbol{J}_{G1} \dot{\boldsymbol{\theta}}_1 \tag{14}$$

$$\dot{\boldsymbol{x}} = \tilde{\boldsymbol{J}} \dot{\boldsymbol{\theta}}_1, \qquad \tilde{\boldsymbol{J}} \triangleq \left(\boldsymbol{J}_1 - \boldsymbol{J}_2 \boldsymbol{J}_{G2}{}^{-1} \boldsymbol{J}_{G1}\right) \in R^{m \times m} \tag{15}$$

The above equations are modified differential forms of Eqs.(1) and (10) and are therefore holonomic in nature. Since the Jacobian $\tilde{\boldsymbol{J}}$ is square, the system in Eq.(15) virtually represents a nonredundant manipulator with $\boldsymbol{x}$ as the workspace and $\boldsymbol{\theta}_1$ as the joint space. Therefore the pseudoinverse control of Eq.(15)

$$\dot{\boldsymbol{\theta}}_1 = \tilde{\boldsymbol{J}}^{\#} \dot{\boldsymbol{x}} \tag{16}$$

imposes no nonholonomic constraints. This follows from our discussion in section 2.1. This along with the fact that Eq.(15) is a holonomic equation implies that the constraints of motion of the system in Eq.(16) are holonomic. Therefore closed paths in the workspace $\boldsymbol{x}$ will result in closed paths in the joint space $\boldsymbol{\theta}_1$, provided the manipulator does not pass through any singular configuration. This follows from our discussion earlier in this section. Additionally, since Eq.(14) is holonomic, closed paths in the space of the independent variables $\boldsymbol{\theta}_1$ will result in closed paths in the space of $\boldsymbol{\theta}_2$ - the dependent variable. In effect the extended Jacobian method will lift closed paths in the workspace $\boldsymbol{x}$ to closed paths in the joint space comprising of both $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$.

2.3 Redundancy in Space Manipulator Systems

Space robots exhibit nonholonomic redundancy [12] - a special type of redundancy that exists in the absence of ordinary kinematic redundancy. Unlike ordinary kinematic redundancy, nonholonomic redundancy manifests itself only after a global motion and cannot be characterized by self-motion manifolds. Inspite of fundamental differences, both redundancies are responsible for nonrepeatable motion of the configuration variables under pseudoinverse control. The nonrepeatability in the configuration variables are a direct manifestation of nonholonomic constraints of motion. In the case of ordinarily redundant terrestrial robots the nonholonomic constraints are imposed by the pseudoinverse control itself, whereas in the case of space robots the nonholonomic constraints are naturally imposed by the conservation of angular momentum.

Consider a space manipulator system with manipulator joint variables $\boldsymbol{\theta}_1 \in R^m$ in a workspace $\boldsymbol{x} \in R^m$. The manipulator is chosen to have as many degrees of freedom as the dimension of the workspace to exhibit the manifestation of nonholonomic redundancy in the absence of ordinary kinematic redundancy. The orientation of the space vehicle on which the space manipulator is mounted is denoted by $\boldsymbol{\theta}_0 \in R^k$. Figure 2 depicts a planar space manipulator for which $m = 2$ and $k = 1$. It can be shown [11], [12] that the direct kinematic relation of the space manipulator is of the form

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_0) \tag{17}$$

which has the structure

$$\dot{\boldsymbol{x}} = \boldsymbol{J}_1 \dot{\boldsymbol{\theta}}_1 + \boldsymbol{J}_0 \dot{\boldsymbol{\theta}}_0 \tag{18}$$

in differential form. The nonholonomic constraint due to angular momentum conservation can be expressed as [11]

$$\dot{\boldsymbol{\theta}}_0 = \boldsymbol{H}(\boldsymbol{\theta}_1)\, \dot{\boldsymbol{\theta}}_1 \tag{19}$$

A complete description of the matrix $\boldsymbol{H} \in R^{k \times m}$ can be found in [11]. Equation (19) can be substituted in Eq.(18) to obtain

$$\dot{\boldsymbol{x}} = \widehat{\boldsymbol{J}}\, \dot{\boldsymbol{\theta}}_1, \qquad \widehat{\boldsymbol{J}} \triangleq (\boldsymbol{J}_1 + \boldsymbol{J}_0 \boldsymbol{H}) \tag{20}$$

where $\widehat{\boldsymbol{J}} \in R^{m \times m}$ is the "generalized Jacobian" [19].

The generalized coordinates of the system include $\boldsymbol{\theta}_1 \in R^m$ and $\boldsymbol{\theta}_0 \in R^k$. The total number of generalized coordinates is $(m + k)$ and the dimension of the workspace is m. The redundancy in the space manipulator system is due to the higher dimension of the generalized coordinates than that of the workspace. The difference in the number of generalized coordinates and the workspace variables is equal to the degree of redundancy in the system. This is similar to ordinary kinematic redundancy in terrestrial manipulators.

A fundamental difference in the redundancy between space manipulators and terrestrial manipulators is that the dimension of the input space is equal to that of the generalized coordinates in the case of terrestrial manipulators whereas for space manipulators, the dimension of the input space is smaller than the dimension of the generalized coordinates. The inputs for terrestrial manipulators are the derivatives of all the generalized coordinates, for space manipulators they comprise of the derivatives of the independent generalized coordinates only, the orientation variables of the space vehicle being the dependent generalized coordinates.

A closer look at Eqs.(19) and (20) point out their similarity to Eqs.(14) and (15). Though the equations look similar, they are fundamentally different because Eqs.(14) and (15) are holonomic while Eqs.(19) and (20) are nonholonomic in nature. Due to this difference, the extended Jacobian method of redundancy control results in repeatable joint motion in ordinarily redundant terrestrial manipulators whereas the pseudoinverse control of the generalized Jacobian [19] results in nonrepeatable joint motion in space manipulators. A more complete discussion on this topic is presented next.

Consider the trajectory control in the three different cases of: (a) a nonholonomically redundant space manipulator, (b) an ordinarily redundant terrestrial manipulator, and (c) an ordinarily redundant terrestrial manipulator using the extended Jacobian. For case (a) the forward kinematics is expressed by Eqs.(17) and (19). Since Eq.(19) is nonholonomic, a closed path in $\boldsymbol{\theta}_1$ space does not imply a closed path in the $\boldsymbol{\theta}_0$ space. This implies from Eq.(17) that a closed path in the joint space of the space manipulator does not result in a closed path in the workspace. For cases (b) and (c) the forward kinematics is simply expressed by Eq.(1) - a position constraint. Therefore a closed path in the joint space will necessarily result in closed trajectories of the end-effector variables. For end-effector trajectory control in case (a), we use the pseudoinverse control law from Eq.(20)

$$\dot{\boldsymbol{\theta}}_1 = \hat{\boldsymbol{J}}^{\#} \dot{\boldsymbol{x}} \tag{21}$$

to plan the joint trajectories. Equation (20) is a nonholonomic equation because it was obtained by substituting a nonholonomic equation, namely Eq.(19) into a holonomic equation, namely Eq.(18). Therefore, while using Eq.(21), closed trajectories in the workspace will generally not result in closed trajectories in the joint space. This can be easily proven by contradiction. Suppose that closed trajectories in $\boldsymbol{x}$ produce closed trajectories in $\boldsymbol{\theta}_1$. We know from the nature of Eq.(19) that closed trajectories in $\boldsymbol{\theta}_1$ do not usually produce closed trajectories in $\boldsymbol{\theta}_0$. This will contradict Eq.(17) that requires the trajectories of $\boldsymbol{\theta}_0$ to be closed for closed trajectories of $\boldsymbol{x}$ and $\boldsymbol{\theta}_1$. For case (b) closed trajectories in the workspace do not result in closed trajectories in the joint space whereas for case (c) they do. This follows straight from our discussion in section 2.2. We conclude this section by summarizing the last result in a tabular form.

	Closed Path in Joint Space implies Closed Path in Workpspace	Closed Path in Workspace implies Closed Path in Joint Space
Space Manipulator	False	False
Redundant Manipulator (Pseudoinverse Control)	True	False
Redundant Manipulator (Extended Jacobian)	True	True

2.4 The LBC is not a Sufficient Condition for Repeatability

Shamir and Yomdin [16] studied the repeatability problem in redundant manipulators and derived a necessary and sufficient condition, the Lie Bracket Condition (LBC), for repeatability. In this section we will show that the LBC of [16] is not a sufficient condition for repeatability when applied to arbitrary extended Jacobians [2]. This contradicts some of the discussion made in [9].

In simple words, the LBC [16] states that repeatability in manipulators is assured if and only if the Lie Bracket of any two column vectors $\boldsymbol{k}_i$ and $\boldsymbol{k}_j$ of the matrix $\boldsymbol{K}$, $\boldsymbol{K}$ being the pseudoinverse of the manipulator Jacobian, is a linear combination of the columns of $\boldsymbol{K}$.

For a manipulator with as many degrees of freedom as the dimension of the workspace, the control matrix $\boldsymbol{K}$ is simply the inverse of the manipulator Jacobian, assuming of course, that the manipulator is not at any singular configuration. The LBC is satisfied for the square and full rank matrix $\boldsymbol{K}$, and indeed, we have repeatable joint motion when the end-effector moves along closed paths.

Since the LBC is always satisfied for square matrices with full rank, Luo and Ahmad [9] extrapolated that repeatability can be achieved by simply converting the rectangular Jacobian of a redundant manipulator into a square matrix by imposing additional independent constraints. They supported their argument with the example of the extended Jacobian [2]. This method achieves repeatability by imposing additional independent constraints that are holonomic. The assertion of Luo and Ahmad [9] is not correct because the rectangular Jacobian may be extended into a square matrix of full rank by imposing nonholonomic constraints as well, and nonholonomic systems do not exhibit repeatability. To understand better, we look back at the expressions of motion of a redundant manipulator under extended Jacobian control [2] and a space manipulator, in sections 2.2 and 2.3 respectively. Specifically, we compare Eqs.(12) and (14) in section 2.2 with Eqs.(18) and (19) in section 2.3. We have seen in section 2.3 that these two systems have structurally identical kinematical equations and constraints but the nature of their constraints are different. Since the space manipulator has nonholonomic constraints, closed paths in the workspace do not result in closed paths in the joint space. This tells us that if the constraint in Eq.(14) were nonholonomic, the pseudoinverse control of the extended Jacobian of the redundant manipulator in section 2.2 would exhibit nonrepeatable joint motion as well.

To illustrate nonrepeatability in redundant manipulators with an extended Jacobian, we consider the manipulator in Fig.1. We assume all the link lengths to be equal to 0.5 units for the sake of simplicity. For this manipulator which has a single degree of redundancy, we impose one nonholonomic constraint

$$\dot{\theta}_3 = \sin(\theta_1 + \theta_3)\,\dot{\theta}_1 + \cos(\theta_2 + \theta_3)\,\dot{\theta}_2$$

The extended Jacobian relation of the manipulator takes the following form

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\sin\theta_1 & -\sin\theta_2 & -\sin\theta_3 \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \\ \sin(\theta_1 + \theta_3) & \cos(\theta_2 + \theta_3) & -1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} \tag{22}$$

Except at the singular points, the pseudoinverse of the extended Jacobian in Eq.(22)

will be equivalent to the inverse, for which the LBC will always be satisfied. However, closed paths in the workspace $\boldsymbol{x}$ will not always result in closed paths in the joint space $\boldsymbol{\theta}$, as seen from Fig.3. Therefore, the assertion made in [9] is not correct.

We wish to make two comments in regard to Fig.3. The abcissae and ordinate in Fig.3 are in different scales. Therefore the circular path of the end-effector looks elliptical. For the same reason, the link lengths seem to vary in different configurations. Also, it may be noted that the manipulator exhibits a limit cycle behavior - the drift in the joint angles of the manipulator decreases as the end-effector repeatedly moves along the closed path. This limit cycle behavior of redundant manipulators will be explained later in section 5.

3. A Necessary Condition for Repeatability

Shamir and Yomdin [16] studied the repeatability problem in redundant manipulators and arrived at a Lie Bracket Condition (LBC) as a necessary and sufficient condition for repeatability. The LBC is by itself a necessary and sufficient condition for the integrability of the distribution associated with the Jacobian pseudoinverse. This comes directly from the statement of Frobenius's Theorem [17]. This was proven separately for the 3-dimensional case in [9], [14]. In essence, Shamir and Yomdin [16] asserted that repeatability can be achieved if and only if the solution to the pseudoinverse control problem is integrable. We do not quite agree with this condition since there exists a weaker necessary condition for repeatability.

Terrestrial redundant manipulators under pseudoinverse control and nonholonomically redundant space manipulators are both constrained systems, and the repeatability problem in these redundant systems is a search for closed trajectories of their configuration variables. We take into consideration the constraints of the systems by searching for closed trajectories of the independent configuration variables that result in closed trajectories of the dependent configuration variables. The change in the dependent configuration variables is expressed as a line integral along the closed path in the space of the independent configuration variables. This line integral may be conviniently expressed as a surface integral using the generalized Stokes' Theorem on a manifold. If D is an oriented manifold of dimension k, and if ω is a $(k-1)$-form on D, then from Stokes' Theorem [1] we have

$$\int_{\partial D} \omega = \int_{D} d\omega \tag{23}$$

where, ∂D is the path of the line integration and is the boundary of the domain D, and $d\omega$ is a differential k-form obtained by exterior differentation of ω. In the case of a planar terrestrial manipulator with three links and a single degree of redundancy, the domain D is a 2-dimensional manifold, and the differential 1-form on D has the functional dependence

$$\omega = d\phi_0 = g_1(\phi_1, \phi_2)\, d\phi_1 + g_2(\phi_1, \phi_2)\, d\phi_2 \tag{24}$$

where, ϕ_0 is the dependent joint variable, and ϕ_1 and ϕ_2 are the independent joint variables of the manipulator. If Eq.(24) were to depict the constraint in a nonholonomically redundant planar space robot with two links, ϕ_0 would represent the orientation of the space vehicle in the plane, and ϕ_1 and ϕ_2 would represent the joint variables of the two link manipulator.

Using Stokes' theorem, the line integration of $d\phi_0$ along a path ∂D on the 2-dimensional manifold D of ϕ_1 and ϕ_2 is expressed as

$$\begin{aligned}\int_{\partial D} d\phi_0 &= \int_{D^\alpha}\left[\frac{\partial g_2}{\partial \phi_1}-\frac{\partial g_1}{\partial \phi_2}\right] d\phi_1 \wedge d\phi_2 \\ &= \pm\int_{D}\left[\frac{\partial g_2}{\partial \phi_1}-\frac{\partial g_1}{\partial \phi_2}\right] d\phi_1 d\phi_2\end{aligned}$$

where "$\wedge$" denotes the exterior product, and α, the orientation of D has the same orientation as $d\phi_1 \wedge d\phi_2$ when the direction along the path is counterclockwise, otherwise α has the same orientation as $d\phi_2 \wedge d\phi_1$.

If the constraint given by Eq.(24) were a holonomic constraint, then we would have

$$\frac{\partial g_2}{\partial \phi_1}=\frac{\partial g_1}{\partial \phi_2} \tag{25}$$

Then the change in the variable ϕ_0 would be zero for all closed paths in the domain D because of the integrable nature of the constraints. This would ensure repeatability. Our contention is that integrability is a sufficient condition for repeatability but is not a necessary condition. For the nonholonomically redundant space manipulator or the terrestrial redundant manipulator the condition given by Eq.(25) does not hold good, yet repeatability can be achieved for certain closed paths in the domain D. We define

$$\frac{\partial g_2}{\partial \phi_1}-\frac{\partial g_1}{\partial \phi_2} \triangleq F(\phi_1,\phi_2) \tag{26}$$

The change in the dependent joint variable ϕ_0 of the redundant manipulator for a positive direction of travel in the space of the independent joint variables is equivalent to

$$\begin{aligned}\int_{\partial D} d\phi_0 &= \int_{D} F(\phi_1,\phi_2)\ d\phi_1 d\phi_2 \\ &= F(\phi_1^*,\phi_2^*)\int_{D} d\phi_1 d\phi_2 = F(\phi_1^*,\phi_2^*)\,A(D)\end{aligned}$$

where, the above equation was obtained by the application of the mean value theorem of integral calculus. The function F is assumed to be continuous in the entire domain D and hence the mean value theorem applies. ϕ_1^* and ϕ_2^* denote some point within the domain D, and $A(D)$ is the measure of the domain D; in this case it is simply equal to the area enclosed within the closed curve ∂D. $F(\phi_1^*,\phi_2^*)$ can also be interpreted as the mean value of the function F, defined in Eq.(26), taken over the domain D. If this mean value happens to be zero, then we would have a zero net change in the dependent joint variable of the redundant manipulator. This would ensure repeatability in the joint motion of the terrestrial manipulator. For the space manipulator this would ensure repeatability in the motion of the end-effector in the workspace. We are now ready to state the necessary condition for the repeatable motion of the redundant manipulator.

Proposition: A necessary condition for the repeatable motion of the redundant manipulator is that the closed path ∂D which is the boundary of the domain D in the independent configuration space should enclose at least one point where the function F defined by Eq.(26) is equal to zero.

The proof of the proposition stated above is quite straightforward and is left to the reader.

If the necessary condition for repeatability is satisfied, it may be possible to find paths in the space of the independent configuration variables such that the net change of the dependent configuration variables is zero over the closed path. The closed path in the independent configuration space will then be like a "holonomic loop" over which the nonholonomic system will exhibit holonomic behavior globally. Incidentally, the holonomic loops will not belong to any integral surface and as such the LBC or the integrability condition, defined in Appendix-A, will not be satisfied at all points along the loop.

4. Repeatability in Nonholonomically Redundant Space Manipulators

There are two different repeatability problems for a space manipulator system: (a) the direct problem of finding a closed path in the joint space of the manipulator such that the end-effector traces a closed path, and (b) the inverse problem of finding a closed path in the workspace that will result in a closed path in the joint space under pseudoinverse control. The inverse problem can be solved by simply solving the direct problem when the number of manipulator joints is equal to the dimension of the workspace, as in our case. This is true because in such situations the pseudoinverse is identical to the inverse assuming that the manipulator is not at any singular configuration. In this section we consider the direct repeatability problem of a planar space robot with two links mounted on a space vehicle as shown in Fig.2. Since the manipulator has two links, the system will exhibit nonholonomic redundancy in the absence of ordinary kinematic redundancy.

The Cartesian coordinates of the end-effector x_E, y_E of the manipulator have a functional dependence of the form

$$x_E = f_1(x_0, y_0, \theta_0, \theta_1, \theta_2), \qquad y_E = f_2(x_0, y_0, \theta_0, \theta_1, \theta_2) \tag{27}$$

where x_0 and y_0 are the coordinates of the center of mass of the space vehicle, θ_0 is the orientation of the vehicle, and θ_1 and θ_2 are the joint variables. The motion of the center of mass of the space vehicle is governed by a holonomic constraint due to linear momentum conservation. For zero initial linear momentum, this can be reduced to the form [11]

$$x_0 = f_3(\theta_0, \theta_1, \theta_2), \qquad y_0 = f_4(\theta_0, \theta_1, \theta_2) \tag{28}$$

Since we are looking into the repeatability problem of a planar space robot, we consider closed trajectories of the joint variables. If the orientation of the space vehicle trace a closed curve as the joints move along a closed trajectory, it is clear from Eqs.(27) and (28) that all the configuration variables including x_0, y_0, x_E, and y_E will move along closed trajectories. This is not true in the general case.

When the joints move along closed trajectories and the system maintains zero angular momentum, the change in the orientation of the space vehicle is expressed as a surface integral using the generalized Stokes' Theorem on a manifold, as given by Eq.(23). The domain D will be the 2-dimensional joint space of the manipulator and the differential 1-form on D will be the constraint due to the conservation of angular momentum, given as

$$\begin{aligned}\omega = d\theta_0 &= g_1(\theta_1, \theta_2)\, d\theta_1 + g_2(\theta_1, \theta_2)\, d\theta_2 \\ &= -\left(\frac{B}{A}\right) d\theta_1 - \left(\frac{C}{A}\right) d\theta_2\end{aligned} \tag{29}$$

where A, B, and C are functions of θ_1 and θ_2 and are defined in Appendix-B. The function F defined in Eq.(26) is therefore equal to

$$F(\theta_1, \theta_2) \triangleq \frac{\partial}{\partial \theta_2}\left(\frac{B}{A}\right) - \frac{\partial}{\partial \theta_1}\left(\frac{C}{A}\right) \tag{30}$$

We now present a simple method to plan repeatable paths for the space manipulator. All paths that will ensure repeatability will have to satisy the necessary condition for repeatability, developed in section 3. Therefore, we first take a look at all points in the θ_1-θ_2 space where the function $F(\theta_1, \theta_2)$ in Eq.(30) is identically zero. The set of all such points constitute a smooth curve, as seen in Fig.4.

We assume our closed path to have an elliptical shape. This path, as seen in Fig.5, can be parameterized as follows:

$$\begin{aligned}\theta_1 &= \theta_{10} + a \cos\phi \cos 2\pi t - b \sin\phi \sin 2\pi t \\ \theta_2 &= \theta_{20} + a \sin\phi \cos 2\pi t + b \cos\phi \sin 2\pi t\end{aligned} \qquad t \in [0, 1] \tag{31}$$

where, a, and b are the major and minor axes of the ellipse, ϕ is the angle of inclination of the ellipse with the θ_1 axis, and θ_{10} and θ_{20} are the coordinates of the center of the ellipse. The velocities of the joints of the manipulator can be easily obtained from the above equation as a function of time. Consequently, the rate of change of the orientation of the space vehicle can be obtained from Eq.(29) as a function of time.

We start with an initial elliptical path which is characterized by the parameters θ_{10}, θ_{20}, a, b, and ϕ. The initial choices of these parameters are quite arbitrary. We only make sure that the elliptical path encompasses at least one point where the function F defined by Eq.(30) is equal to zero. This condition can be easily satisfied by considering Fig.4 which provides the set of all points where the function F vanishes.

Our goal is now to change the five parameters of the ellipse so that the surface integral of the function F in Eq.(30) over the elliptical path is equal to zero. Of the five different parameters a and b are not allowed to change independent of one another. This is because we want to eliminate the trivial solution where the surface integral is zero because the area of the closed path is equal to zero. One simple way to avoid this situation is to impose the restriction that the area of the ellipse is a constant. This is equivalent to the constraint

$$a\, db + b\, da = 0 \tag{32}$$

We define a function V as follows

$$V = \zeta^2, \qquad \zeta \triangleq \iint_D F(\theta_1, \theta_2)\, d\theta_1 d\theta_2 \tag{33}$$

and solve the unconstrained minimization problem by implicitly assuming that a and b are dependent. In Eq.(33) ζ is equal to the net change in the orientation of the space vehicle as the joint variables move along closed paths. While there are many methods for unconstrained minimization, we choose the simplest method of steepest descent [20]. Other alternative methods that can be used are the conjugate direction method by Fletcher and Reeves [4], and the variable metric method [20] that offer improvement over the method of steepest descent. In our case the method of steepest descent works well and therefore we adopted it only for its simplicity.

The correct choice of the independent parameters θ_{10}, θ_{20}, ϕ, and a that provied us with the steepest direction of descent of the function V are computed as

$$d\theta_{10} = -\zeta \frac{\partial \zeta}{\partial \theta_{10}}, \qquad d\theta_{20} = -\zeta \frac{\partial \zeta}{\partial \theta_{20}}, \qquad d\phi = -\zeta \frac{\partial \zeta}{\partial \phi}, \qquad da = -\zeta \frac{\partial \zeta}{\partial a}$$

In the above equation, the quantities $(\partial\zeta/\partial\theta_{10})$, $(\partial\zeta/\partial\theta_{20})$, $(\partial\zeta/\partial\phi)$, and $(\partial\zeta/\partial a)$ are computed by numerical partial differentiation. While computing the term $(\partial\zeta/\partial a)$ it has to be remembered that a change in a is accompanied by a change in b given by the constraint in Eq.(32).

The optimization technique discussed above provides us with a systematic way to reach the local minimum value of the function V. If this minimum value is zero, then we have converged upon the desired path around which the space robot will exhibit pseudo-holonomic behavior. In the general case, the method of steepest descent does not guarantee the convergence of a function to its global minimum value. However, in our case the method always converged to the global minimum value of $V = 0$, because of the particular nature of the function F in Eq.(30).

5. Repeatable Motion under Pseudoinverse Control

The repeatability problem in redundant manipulators under pseudoinverse control is a search for closed trajectories of the end-effector that result in closed joint trajectories. Since the pseudoinverse control is actually a form of nonholonomically constrained motion, closed joint trajectories can be obtained only if the dependent joint variables move along a closed path as the independent joint variables do. Our logical first step is therefore to search for closed trajectories of the independent joint variables that result in closed trajectories of the dependent joint variables.

To further our discussion, we consider the planar redundant manipulator in Fig.1 with unity link lengths. This example was considered in [7], [9], and [16]. The kinematic relations of this manipulator are given in Eq.(4) and its Jacobian is given by Eq.(5). The nonholonomic constraint of the manipulator under pseudoinverse control is given by Eq.(6), which is not of the form given by Eq.(24). To reduce it to this form, we use the simple transformation

$$\psi_1 = \theta_1, \quad \psi_2 = \theta_2 - \theta_1, \quad \psi_3 = \theta_3 - \theta_2 \tag{34}$$

The transformed constraint equation

$$[\sin(\psi_2 + \psi_3) - \sin\psi_2 - \sin\psi_3]\, d\psi_1 - \sin\psi_2\, d\psi_3 + [\sin(\psi_2 + \psi_3) - \sin\psi_2]\, d\psi_2 = 0 \tag{35}$$

is then of the form as given by Eq.(24). Under the assumption that

$$\sin\psi_2 + \sin\psi_3 - \sin(\psi_2 + \psi_3) \neq 0 \tag{36}$$

the change in the dependent variable ψ_1, as the independent variables ψ_2 and ψ_3 move along a closed path, can be shown to be

$$\int_{\partial D} d\psi_1 = \iint_D \frac{1}{[\sin\psi_2 + \sin\psi_3 - \sin(\psi_2 + \psi_3)]}\, d\psi_2 d\psi_3$$

using Stokes' Theorem [1]. The function F defined by Eq.(26) is given as

$$F(\psi_2, \psi_3) \triangleq \frac{1}{[\sin\psi_2 + \sin\psi_3 - \sin(\psi_2 + \psi_3)]} \tag{37}$$

and is not equal to zero anywhere in the ψ_2-ψ_3 plane. The necessary condition for repeatability is therefore not satisfied. This means that when the condition in Eq.(36) holds good, the redundant manipulator cannot exhibit repeatability. The condition in Eq.(36) does not hold good when we have any one of the three cases

$$\begin{array}{llcl} (a) & \psi_2 = 0 & \Longleftrightarrow & \theta_1 = \theta_2 \\ (b) & \psi_3 = 0 & \Longleftrightarrow & \theta_2 = \theta_3 \\ (c) & \psi_2 + \psi_3 = 0 & \Longleftrightarrow & \theta_3 = \theta_1 \end{array}$$

Using Eq.(35), it is possible to show that the three cases above imply

$$\begin{array}{llcl} (a) & \psi_2 = 0 & \longmapsto & d\psi_2 = 0 \\ (b) & \psi_3 = 0 & \longmapsto & d\psi_3 = 0 \\ (c) & \psi_2 + \psi_3 = 0 & \longmapsto & d(\psi_2 + \psi_3) = 0 \end{array}$$

Therefore, each of the three cases represent an integral surface. These results are identical to that obtained by Shamir and Yomdin [16] using the Lie Bracket Condition (LBC). We have to agree that for the particular example considered, repeatability can be achieved only if the LBC holds good, *i.e.* the LBC is a necessary condition for repeatability.

The LBC is not a necessary condition for repeatability in the general case. To illustrate this concept we consider the same manipulator as in Fig.1, but we redefine the configuration variables. Once again we assume the link lengths to be unity for the sake of simplicity. If the new configuration variables are ψ_1, ψ_2, and ψ_3, as defined by Eq.(34), then the kinematic relations of the manipulator are

$$\begin{aligned} X &= \cos\psi_1 + \cos\psi_{12} + \cos\psi_{123} \\ Y &= \sin\psi_1 + \sin\psi_{12} + \sin\psi_{123} \end{aligned} \tag{38}$$

and the manipulator Jacobian is given by

$$\boldsymbol{J}_\psi = \begin{pmatrix} -\sin\psi_1 - \sin\psi_{12} - \sin\psi_{123} & -\sin\psi_{12} - \sin\psi_{123} & -\sin\psi_{123} \\ \cos\psi_1 + \cos\psi_{12} + \cos\psi_{123} & \cos\psi_{12} + \cos\psi_{123} & \cos\psi_{123} \end{pmatrix} \tag{39}$$

where we used the compact notation ψ_{12} and ψ_{123} to denote $(\psi_1 + \psi_2)$ and $(\psi_1 + \psi_2 + \psi_3)$ respectively. We note that the Jacobian $\boldsymbol{J}_\psi$ in the above equation is quite different from the Jacobian $\boldsymbol{J}_\theta$ in Eq.(5). Consequently, the pseudoinverse control for $\boldsymbol{J}_\psi$ would be considerably different from that of $\boldsymbol{J}_\theta$.

The nonholonomic constraint of the manipulator under pseudoinverse control of $\boldsymbol{J}_\psi$ is found to be

$$\sin\psi_3\, d\psi_1 - [\sin\psi_3 + \sin(\psi_2 + \psi_3)]\, d\psi_2 + [\sin\psi_2 + \sin(\psi_2 + \psi_3)]\, d\psi_3 = 0 \tag{40}$$

Assuming $\sin\psi_3 \neq 0$, the change in the dependent variable ψ_1, as the independent variables ψ_2 and ψ_3 move along closed paths, can be shown to be

$$\int_{\partial D} d\psi_1 = \iint_D \frac{\sin\psi_3 \cos\psi_2 + \sin\psi_3 \cos(\psi_2 + \psi_3) - \sin\psi_2}{\sin\psi_3{}^2}\, d\psi_2 d\psi_3 \tag{41}$$

The function F defined in Eq.(26) is therefore equal to

$$F(\psi_2, \psi_3) \triangleq \frac{\sin\psi_3 \cos\psi_2 + \sin\psi_3 \cos(\psi_2 + \psi_3) - \sin\psi_2}{\sin\psi_3{}^2} \tag{42}$$

To plan repeatable paths for the redundant manipulator under pseudoinverse control, the necessary condition for repeatability discussed in section 3 has to be satisfied. Therefore, we take a look at a set of points where the function $F(\psi_2, \psi_3)$ vanishes. A set of these points is given in Fig.6. We assume the closed path in the space of the independent joint variables, ψ_2 and ψ_3, to have an elliptical shape, as shown in Fig.7. The path in ψ_2 and ψ_3 is parameterized in a way similar to Eq.(31). We start with an initial choice of the parameters ψ_{20}, ψ_{30}, a, b, and ϕ. The choice is quite arbitrary except for the fact that the path should enclose at least one point where the function F defined by Eq.(42) is equal to zero. This condition can be satisfied by considering Fig.6. Our goal is now to change the five parameters of the elliptical path so that the surface integral of the function F in Eq.(42) over the path is equal to zero. We achieve our goal by adopting the optimization technique outlined in section 4. The optimized path can then be likened to a "holonomic loop" over which the joint angles of the redundant manipulator will exhibit holonomic behavior globally.

Let us define the optimized closed path in the joint space to be $\mathcal{C_J}$. Using forward kinematics, as given by Eq.(38), we can obtain a closed path in the workspace $\mathcal{C_W}$ of the redundant manipulator from $\mathcal{C_J}$. Under pseudoinverse control, $\mathcal{C_W}$ will map back to $\mathcal{C_J}$

in the joint space. This is true because (a) the mapping $\mathcal{C}_J \longmapsto \mathcal{C}_W$ satisfies the kinematic relations of the manipulator, and (b) the closed path $\mathcal{C}_J$ satisfies the nonholonomic constraint due to pseudoinverse control.

Before we conclude this section we wish to make two comment:

1. It can be shown that the Lie Bracket Condition (LBC) in [16] is not satisfied at every point on $\mathcal{C}_J$, yet the net change in the dependent joint variable ψ_1 along $\mathcal{C}_J$ is zero. Therefore, we can achieve repeatability in the absence of an integral surface.
2. Our second comment is in regards to the limit cycle behavior in redundant manipulators. It was noted in [7] that under pseudoinverse control some manipulators drift continuously while others exhibit limit cycle behavior. Our studies lead us to believe that the drift in a redundant manipulator may be self-optimizing in the sense that the drift may decrease with every cycle of end-effector motion. When such a situation arises, the drift finally goes to zero and the manipulator reaches a limit cycle.

6. Simulations

6.1 A Nonholonomically Redundant Space Manipulator

We carried out several computer simulations. Here we present results of one particular case. The kinematic and dynamic parameters of the planar space robot were chosen to be

Kinematic and Dynamic parameters

	Mass (kg)	Inertia (kg-m^2)	Length (m)
Vehicle	27.440	1.520	$r = 0.20$
Link-1	5.380	0.115	$l_1 = 0.50$
Link-2	2.640	0.028	$l_2 = 0.35$

The initial parameters of the elliptical path were arbitrarily chosen as

$$a = 1.50000, \quad b = 1.00000, \quad \phi = 0.75000, \quad \theta_{10} = 0.50000, \quad \theta_{20} = 0.50000 \tag{43}$$

where the units are in radians. For these set of values, the numerical value of the surface integral ζ was found to be $\zeta = -0.162775$. The convergence criterion was set at $|\zeta| \leq 1.0 \times 10^{-8}$. The values of the path parameters after convergence were

$$a = 1.31117, \quad b = 1.14381, \quad \phi = 0.79302, \quad \theta_{10} = 0.34094, \quad \theta_{20} = -0.07054 \tag{44}$$

The two elliptical paths are shown in Fig.8. Ellipse I corresponds to the initial choice of the path parameters given by Eq.(43) for which the value of $\zeta = -0.162775$. Ellipse II corresponds to the optimized values of the path parameters given by Eq.(44) and the value of ζ for this path was $\zeta = -9.9636 \times 10^{-9}$. The sinusoidal curve in Fig.4 is inset in Fig.8. This curve passes through both paths I and II and therefore these paths satisfy the necessary condition discussed in section 3.

Figures 9 and 10 depict the motion of the end-effector of the space robot for 20 cycles for the elliptical paths *I* and *II* respectively. The end-effector configuration is seen to drift in Fig.9 but has negligible drift for the closed path in Fig.10. The magnitude of the drift was computed to be approximately 76.96 mm/cycle in the case of path *I* whereas it was only 0.87 mm/cycle for path *II*.

6.2 An Ordinarily Redundant Terrestrial Manipulator

We considered the simple case of a planar three link manipulator with one degree of redundancy, as shown in Fig.1. The kinematic relations of the manipulator and its Jacobian are given by Eqs.(38) and (39). We assumed the links of the manipulator to have equal lengths of 0.5 metres.

The initial parameters of the elliptical path was arbitrarily chosen as

$$a = 1.00000, \quad b = 1.00000, \quad \phi = 0.00000, \quad \psi_{20} = 0.75000, \quad \psi_{30} = 1.50000 \tag{45}$$

where the units are in radians. For these set of values, the numerical value of the surface integral ζ was found to be $\zeta = 1.9838625$. The convergence criterion was set at $|\,\zeta\,| \leq 1.0 \times 10^{-8}$. The values of the path parameters after convergence were

$$a = 1.03337, \quad b = 0.96769, \quad \phi = 0.00232, \quad \psi_{20} = 0.51512, \quad \psi_{30} = 1.42478 \tag{46}$$

The two elliptical paths are shown in Fig.11. Ellipse *I* corresponds to the initial choice of the path parameters given by Eq.(45) for which the value of $\zeta = 1.9838625$. Ellipse *II* corresponds to the optimized values of the path parameters given by Eq.(46) and the value of ζ for this path was $\zeta = 9.9431 \times 10^{-9}$. The sinusoidal curve in Fig.6 is inset in Fig.11. This curve passes through both paths *I* and *II* and therefore these paths satisfy the necessary condition discussed in section 3.

Path *II* in Fig.11 is the optimized path in the ψ_2-ψ_3 plane of the manipulator that results in closed loop motion of the dependent joint variable ψ_1. For an initial value of $\psi_1 = 0.0$, the closed end-effector trajectory $C_{\mathcal{W}}$ that is obtained from these closed joint trajectories $C_{\mathcal{J}}$ is shown in Fig.12. Figure 12 also shows the link configuration of the manipulator at six different points along the trajectory. The joint trajectories obtained through pseudoinverse control of the closed end-effector trajectory in Fig.12 are shown in Fig.13. The joint trajectories in Fig.13 pertain to 5 cycles of end-effector motion. The numerical simulation was continued for more than 100 cycles and the joints were seen to have exceptional repeatability. This conforms to our discussion in section 5.

We wish to make two comments at this juncture:

1. The Lie Bracket Condition (LBC) in [16] is essentially a test for integrability of the distribution spanned by the column vectors of the Jacobian pseudoinverse. Though the LBC can be verified from the Jacobian transpose instead of the pseudoinverse [9], [14], it may still involve a significant amount of symbolic computation. An easier way out is to test the integrability of the constraint imposed by the pseudoinverse control.

We can use Eqs.(7) and (8) to test the integrability of the pseudoinverse constraint In Eq.(40). For the joint trajectories in Fig.13 it can be shown that the condition for integrability is not satisfied at all points along the path, *i.e.* $F(\psi_2, \psi_3)$ in Eq.(42) is not zero at all points along the path. Therefore repeatability is achieved in the absence of integrability. This refutes the LBC in [16].

2. Since the end-effector trajectory is generated from the joint trajectories and since the initial configuration of the dependent joint variable, ψ_1 in our case, is completely arbitrary, any rotation of the end-effector trajectory in Fig.12 about the z-axis will also produce repeatable joint motion. Clearly, there are infinite end-effector trajectories that produce repeatable joint motion.

7. Conclusion

In this paper we promoted the concept that integrability is not a necessary condition for repeatability in nonholonomic systems. This allows us to plan repeatable trajectories for free-flying space manipulators with zero initial momentum whose constraint due to the conservation of angular momentum is not integrable. This is important because it allows a space manipulator to perform repeated tasks in space without any drift in its configuration variables. For terrestrial manipulators under pseudoinverse control the nonholonomic constraint is imposed by the control law. We showed that under pseudoinverse control, repeatability of the joint variables can be achieved in the absence of any integral surface and by virtue of the presence of "holonomic loops". These loops, when they exist, allow a nonholonomic system to exhibit repeatability in its configuration variables. In this paper we presented a simple optimization technique for planning repeatable trajectories for both nonholonomically redundant space manipulators and ordinarily redundant terrestrial manipulators.

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APPENDIX-A

The necessary and sufficient condition that the differential constraint in n variables

$$v_1\,dx_1 + v_2\,dx_2 + \cdots + v_n\,dx_n = 0$$

is integrable, is that the set of equations

$$v_\nu \left(\frac{\partial v_\mu}{\partial x_\lambda} - \frac{\partial v_\lambda}{\partial x_\mu} \right) + v_\mu \left(\frac{\partial v_\lambda}{\partial x_\nu} - \frac{\partial v_\nu}{\partial x_\lambda} \right) + v_\lambda \left(\frac{\partial v_\nu}{\partial x_\mu} - \frac{\partial v_\mu}{\partial x_\nu} \right) = 0$$
$$(\lambda, \mu, \nu = 1, 2, \cdots, n)$$

are satisfied simultaneously for all different combinations of λ, μ and ν [5].

APPENDIX-B

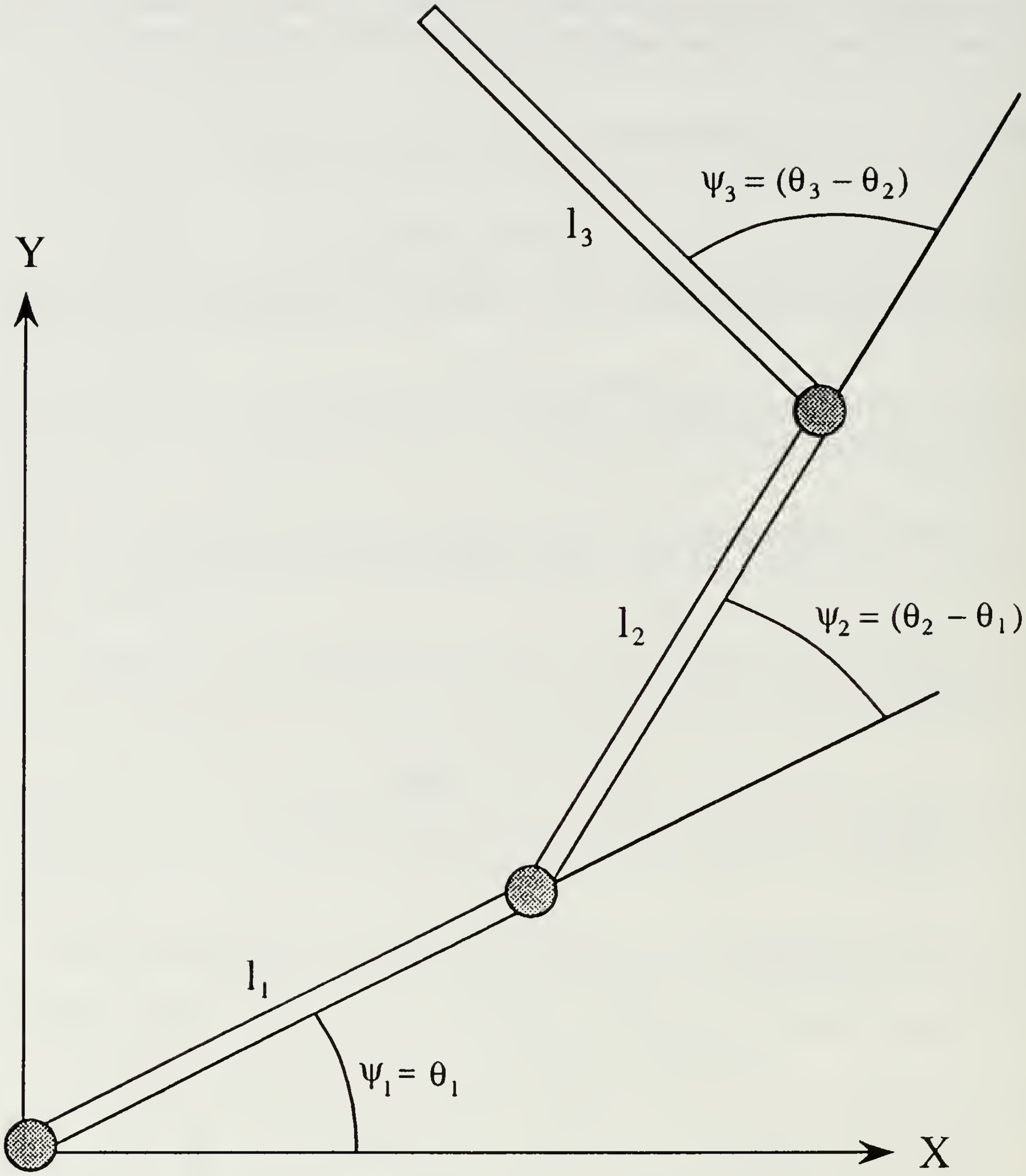
The terms A, B, and C in Eq.(29) are defined as follows

$$\begin{aligned} A \triangleq I_t &+ \frac{1}{M} r^2 m_0(m_1 + m_2) + \frac{l_1^2}{4M}(m_0 m_1 + m_1 m_2 + 4 m_0 m_2) + \frac{l_2^2}{4M} m_2(m_0 + m_1) \\ &+ \frac{1}{M} m_0(m_1 + 2m_2) r l_1 cos\,\theta_1 + \frac{1}{M} m_2(m_0 + 0.5 m_1) l_1 l_2 cos\,\theta_2 + \frac{1}{M} m_0 m_2 r l_2 cos(\theta_1 + \theta_2) \end{aligned}$$

$$\begin{aligned} B \triangleq I_1 + I_2 &+ \frac{l_1^2}{4M}(m_0 m_1 + m_1 m_2 + m_0 m_2) + \frac{l_2^2}{4M} m_2(m_0 + m_1) + \frac{2}{M} m_0(m_1 + 2m_2) r l_1 cos\,\theta_1 \\ &+ \frac{1}{M} m_2(m_0 + 0.5 m_1) l_1 l_2 cos\,\theta_2 + \frac{1}{2M} m_0 m_2 r l_2 cos(\theta_1 + \theta_2) \end{aligned}$$

$$C \triangleq I_2 + \frac{l_2^2}{4M} m_2(m_0 + m_1) + \frac{1}{2M} m_0 m_2 l_1 l_2 cos\,\theta_2 + \frac{1}{2M} m_0 m_2 r l_2 cos(\theta_1 + \theta_2)$$

where, m_0, m_1, and m_2 are the masses of the space vehicle and the two links of the manipulator, I_0, I_1, and I_2 are the moment of inertias of the space vehicle and the two links about their center of masses, r is the distance of the first joint from the center of mass of the vehicle, l_1 and l_2 are the lengths of the two links, $M \triangleq m_0 + m_1 + m_2$, and $I_t \triangleq I_0 + I_1 + I_2$.



$\psi_3 = (\theta_3 - \theta_2)$
l_3
Y
l_2
$\psi_2 = (\theta_2 - \theta_1)$
l_1
$\psi_1 = \theta_1$
X

Figure 1. A planar three link redundant manipulator.

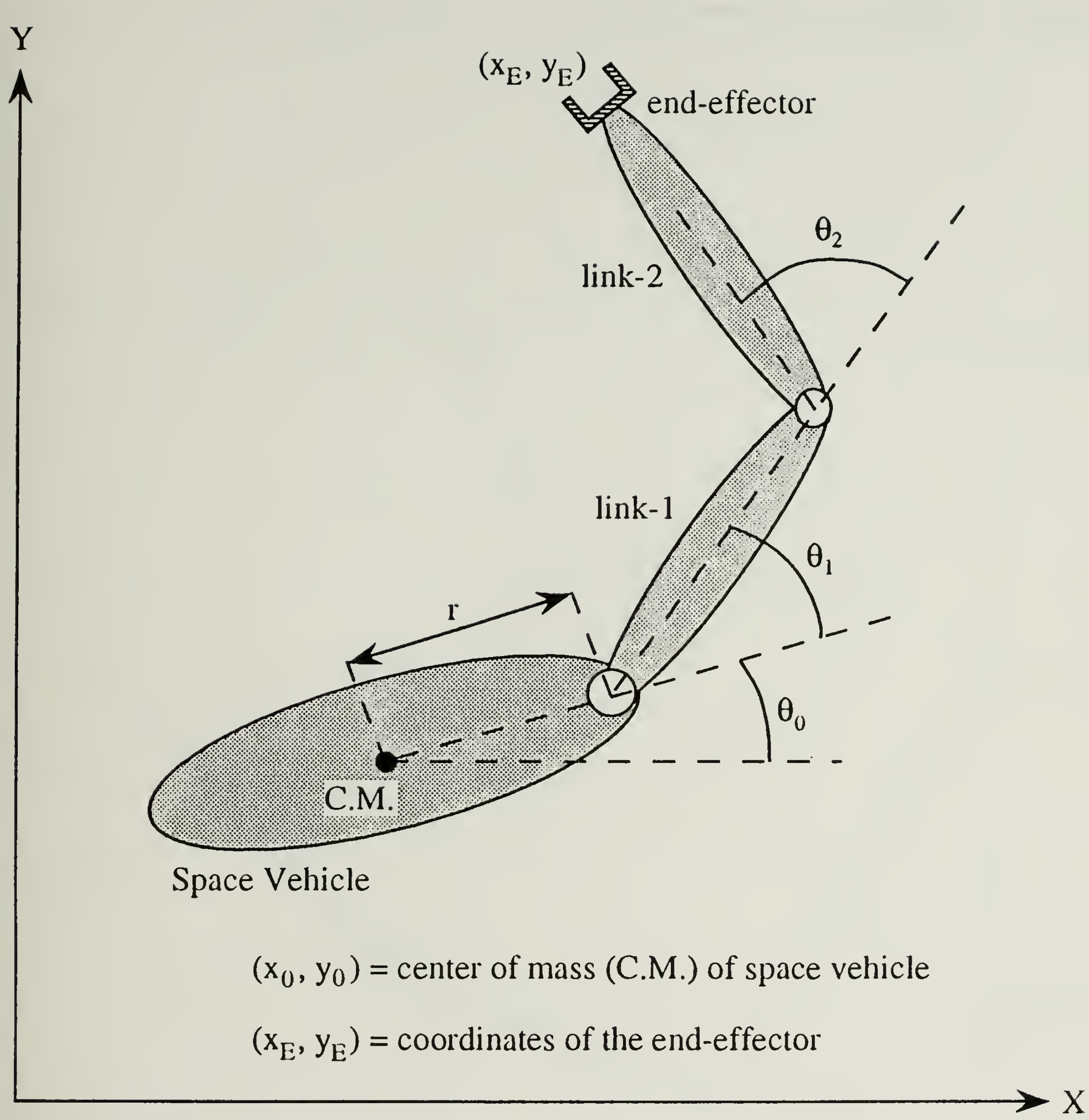

Y
(x_E, y_E)
end-effector
θ_2
link-2
link-1
θ_1
r
θ_0
C.M.
Space Vehicle
(x_0, y_0) = center of mass (C.M.) of space vehicle
(x_E, y_E) = coordinates of the end-effector
X

Figure 2. A two link planar space manipulator mounted on a space vehicle.

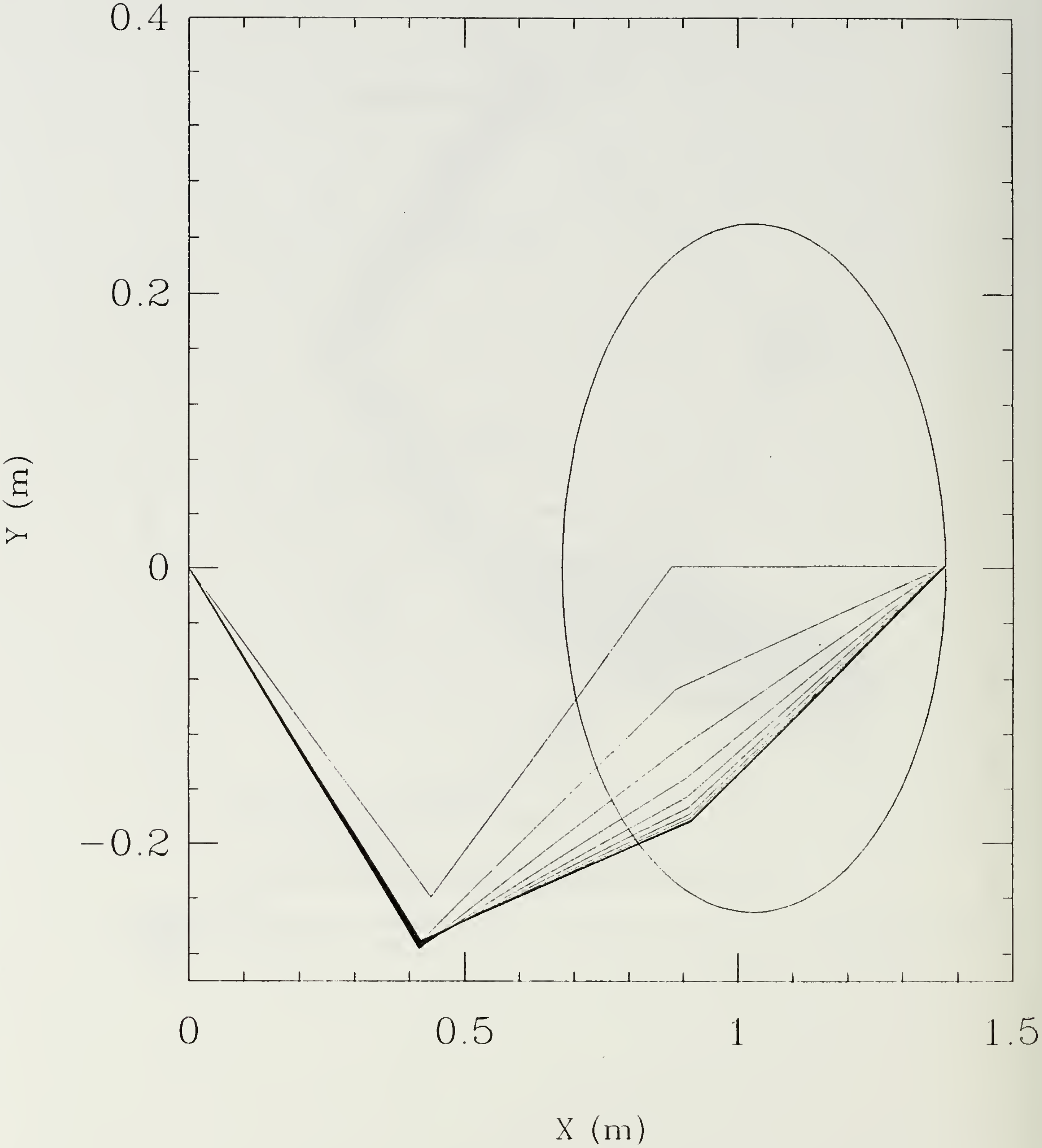


Figure 3. Nonrepeatable joint motion of the three link redundant manipulator under pseudoinverse control of its extended Jacobian.

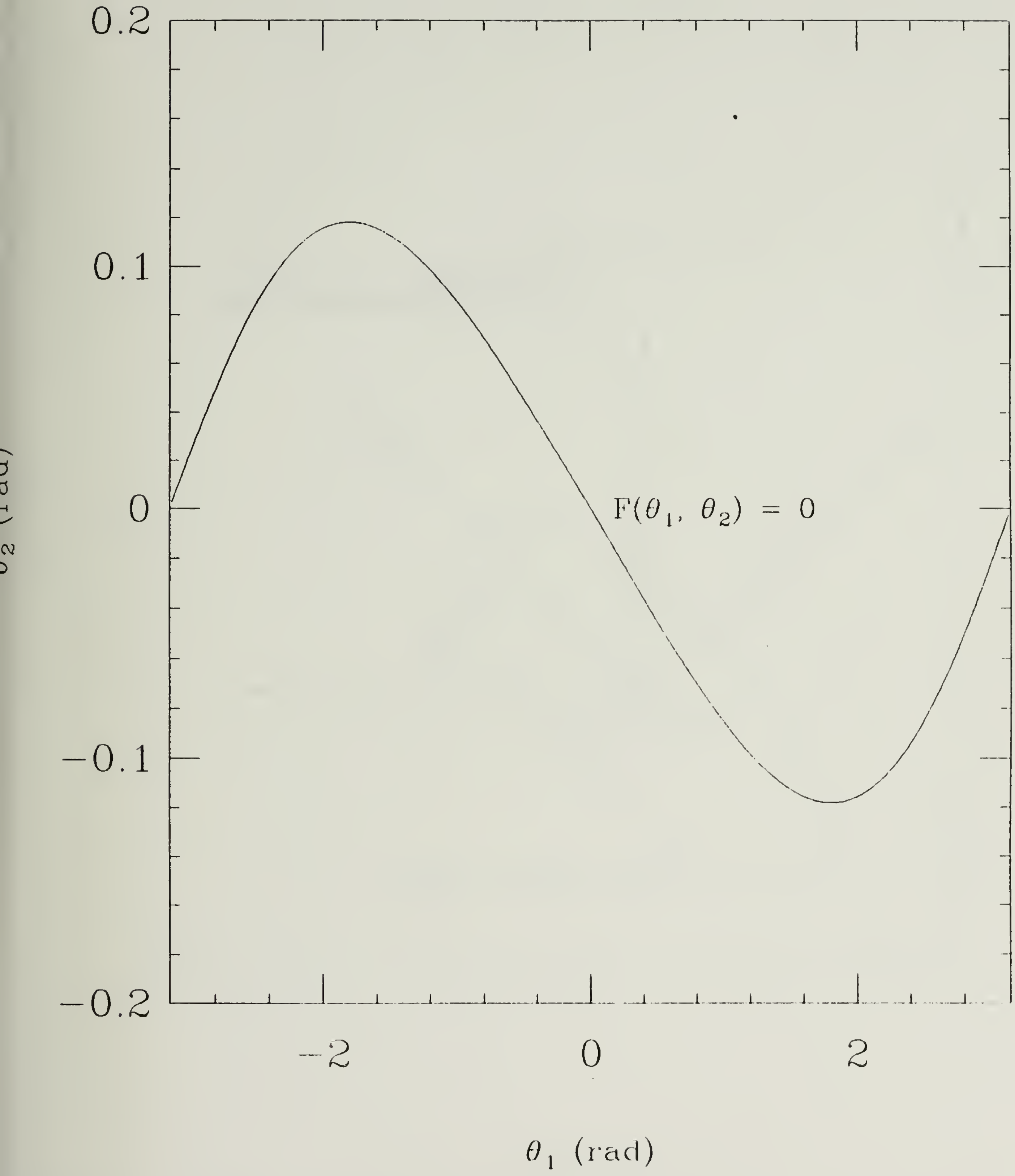


Figure 4. The locus of points in the θ_1-θ_2 plane of the planar space robot where $F(\theta_1, \theta_2) = 0$.

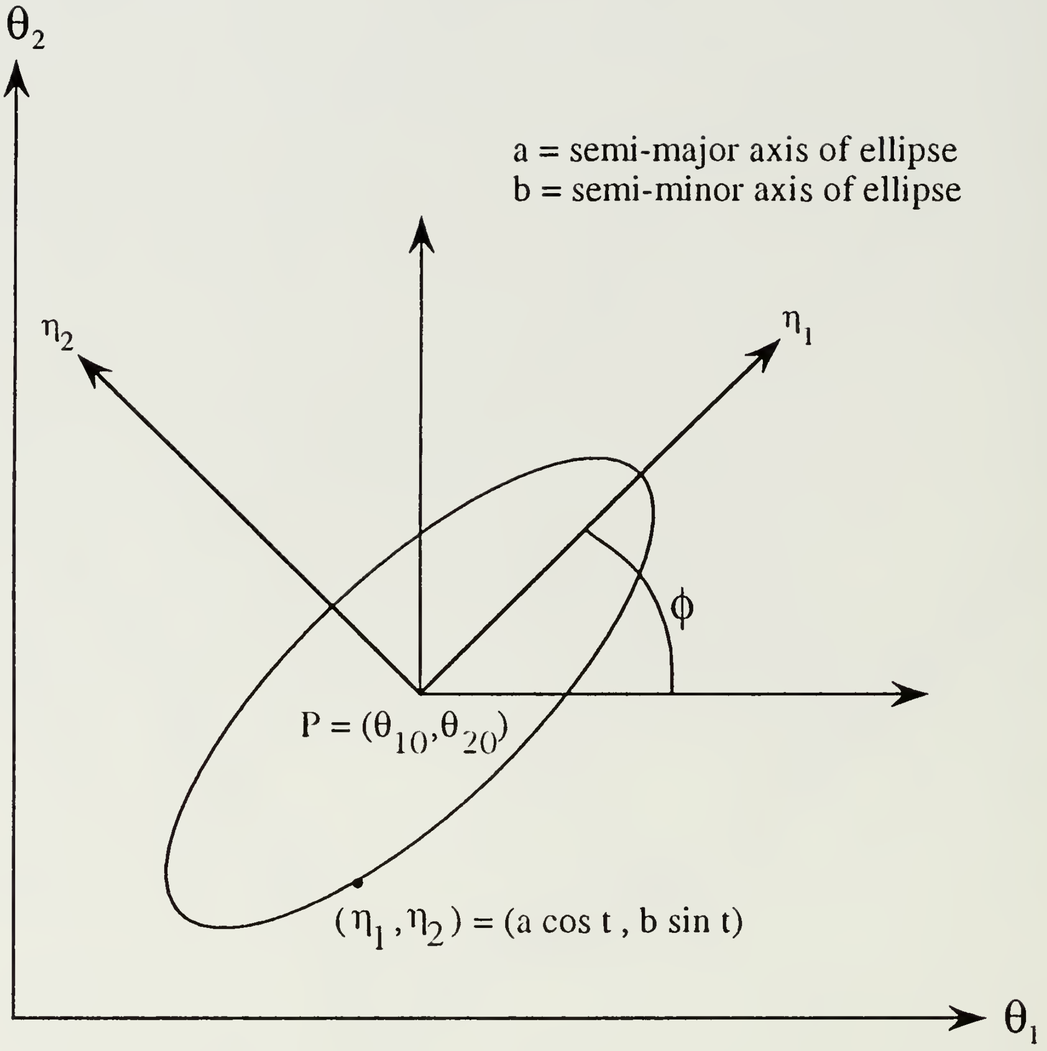

θ_2
a = semi-major axis of ellipse
b = semi-minor axis of ellipse
η_2
η_1
ϕ
$P = (\theta_{10}, \theta_{20})$
$(\eta_1, \eta_2) = (a \cos t, b \sin t)$
θ_1

Figure 5. Parametric representation of the elliptical path in the joint space of the space robot. P is the center of the ellipse, and ϕ is the angle between the major axis of the ellipse and θ_1.

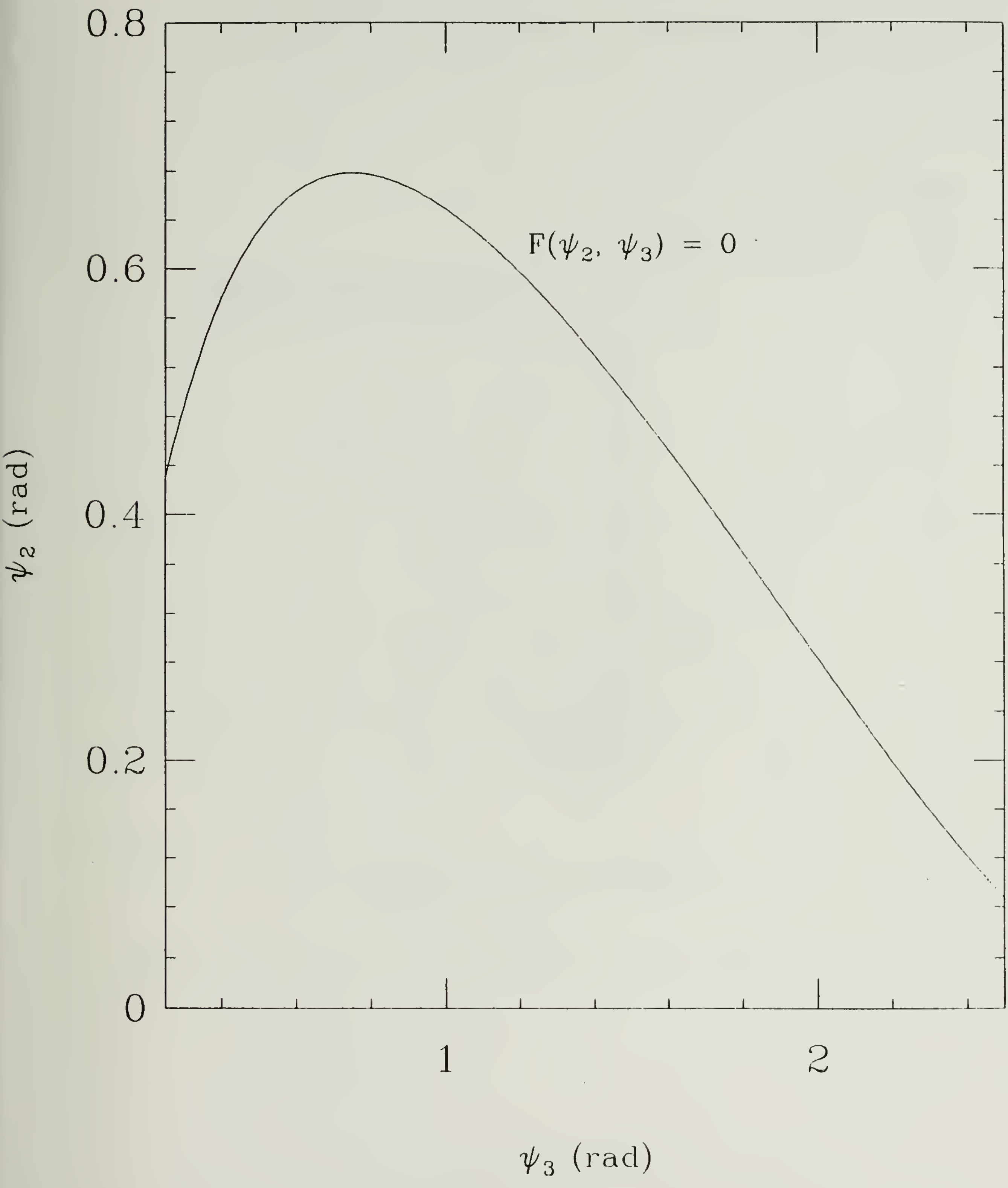
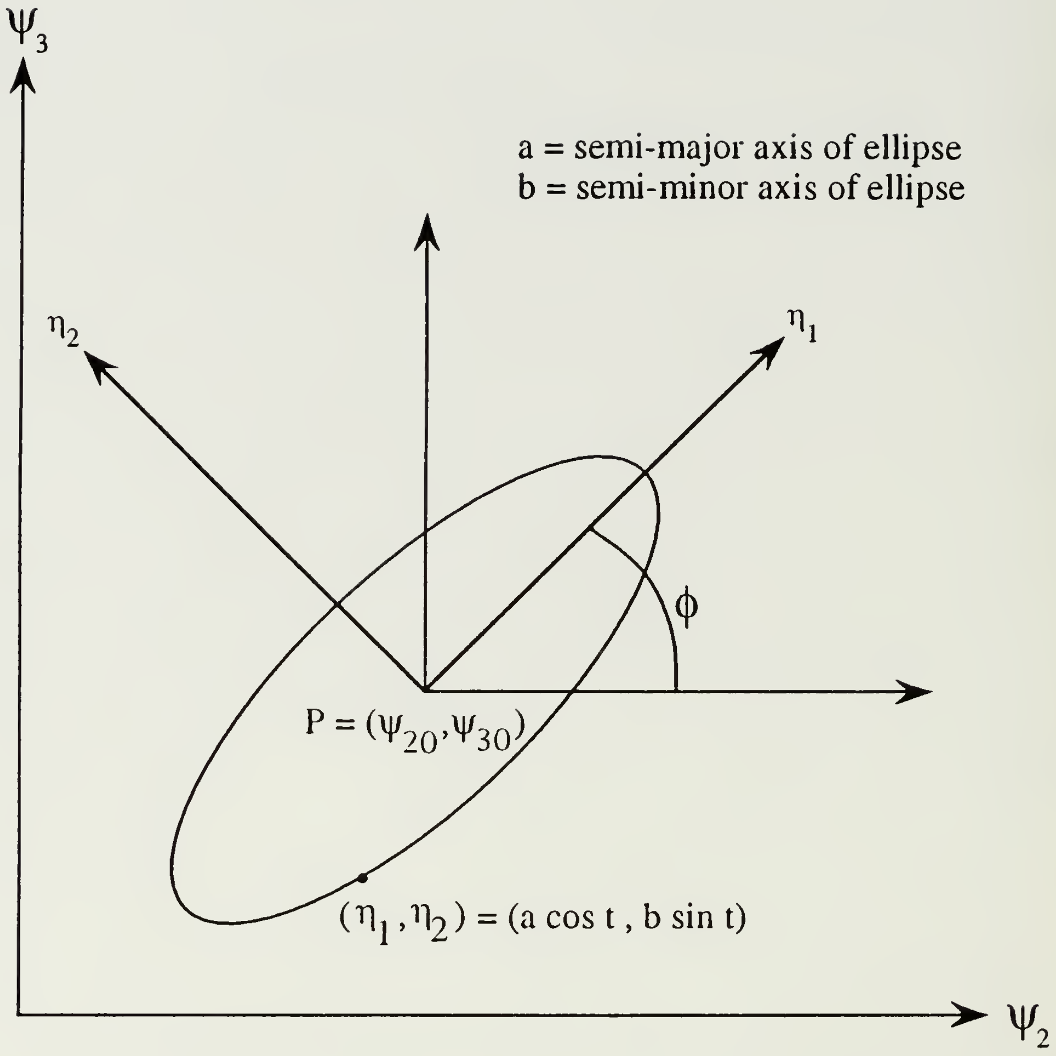


Figure 6. The locus of points in a certain region of the ψ_2-ψ_3 plane of the three link redundant manipulator where $F(\psi_2, \psi_3) = 0$.



ψ_3
a = semi-major axis of ellipse
b = semi-minor axis of ellipse
η_2
η_1
ϕ
$P = (\psi_{20}, \psi_{30})$
$(\eta_1, \eta_2) = (a \cos t, b \sin t)$
ψ_2

Figure 7. Parametric representation of the elliptical path in the space of the independent joint variables of the redundant manipulator. P is the center of the ellipse whose major axis subtends an angle ϕ with ψ_2.

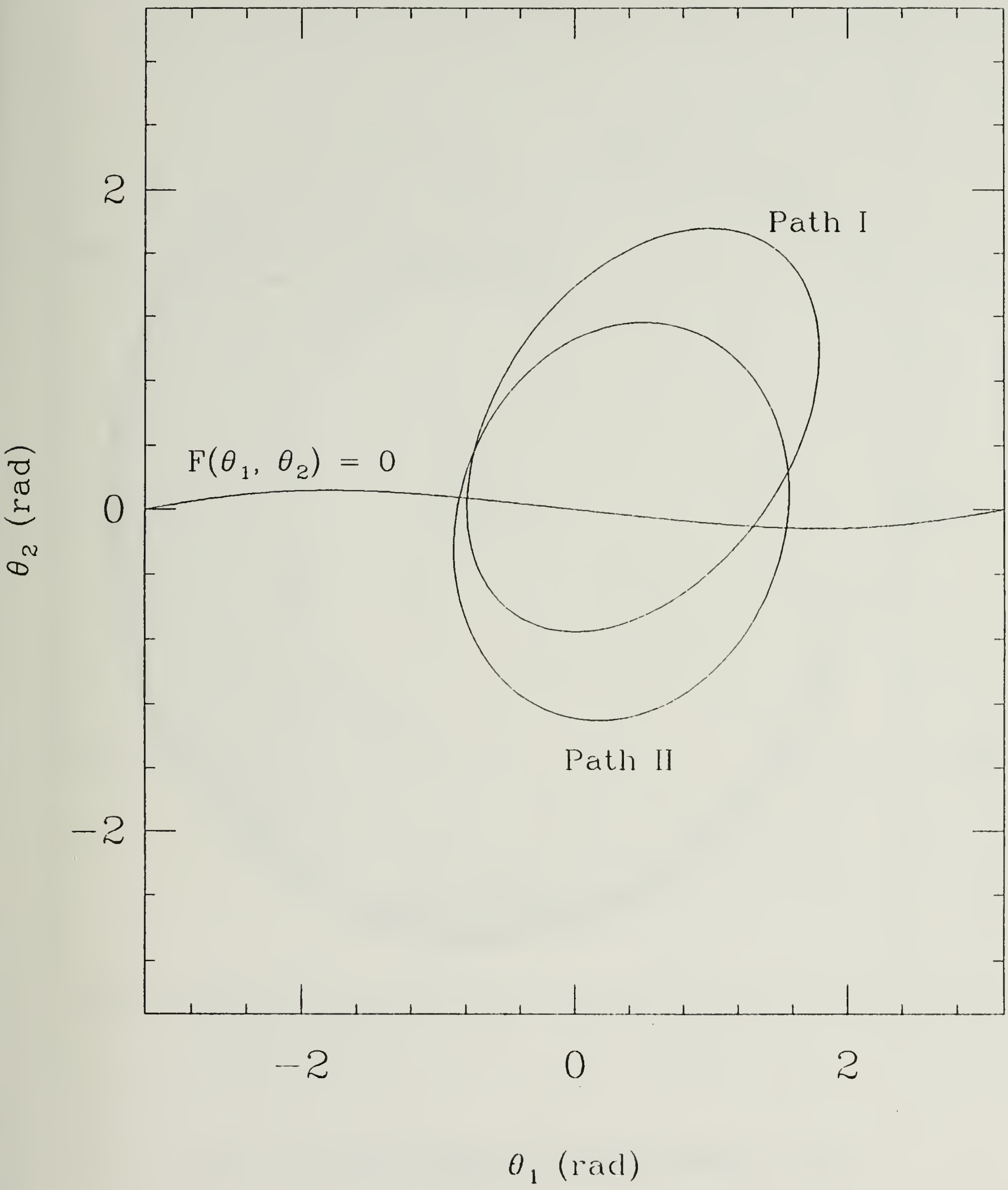


Figure 8. Elliptical paths in the joint space of the planar space robot. Path I is the initially chosen path and Path II is the optimized path.

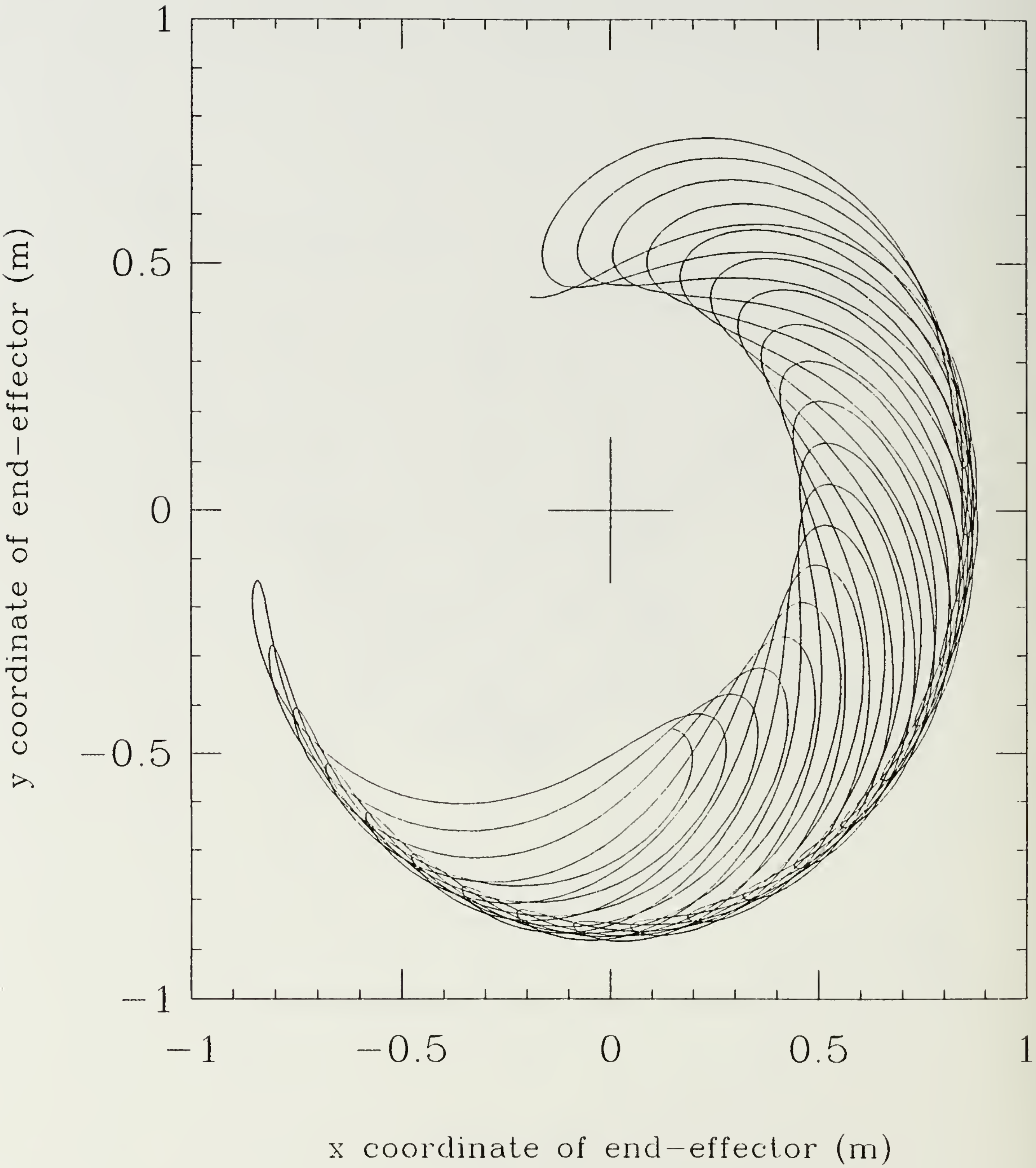


Figure 9. End-effector drift in 20 cycles for Path I in the joint space of the space robot.

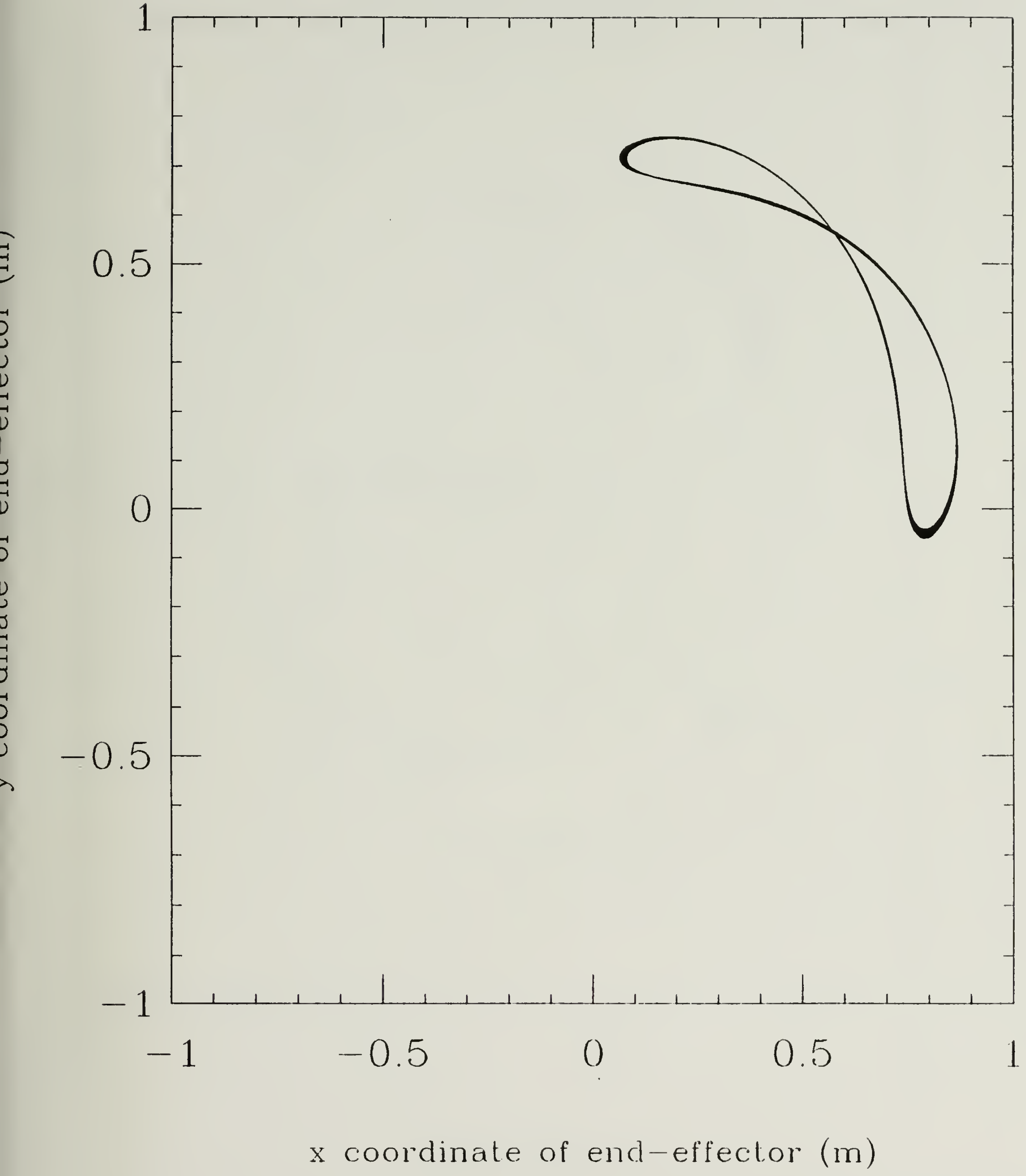


Figure 10. Repeatable end-effector motion for Path II in the joint space of the space robot.

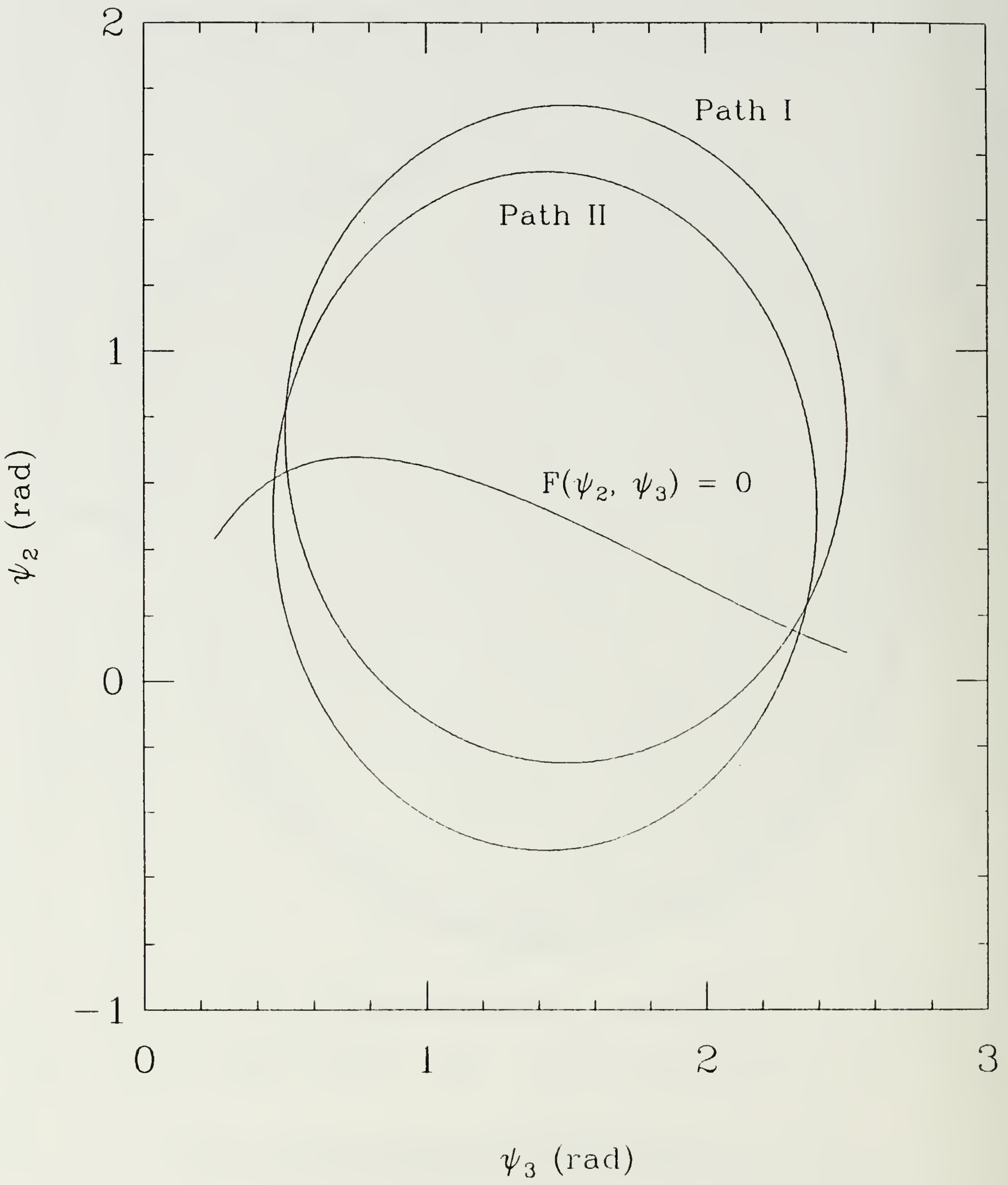


Figure 11. Elliptical paths in the space of the independent joint variables of the three link redundant robot. Path I is the initially chosen path and Path II is the optimized path.

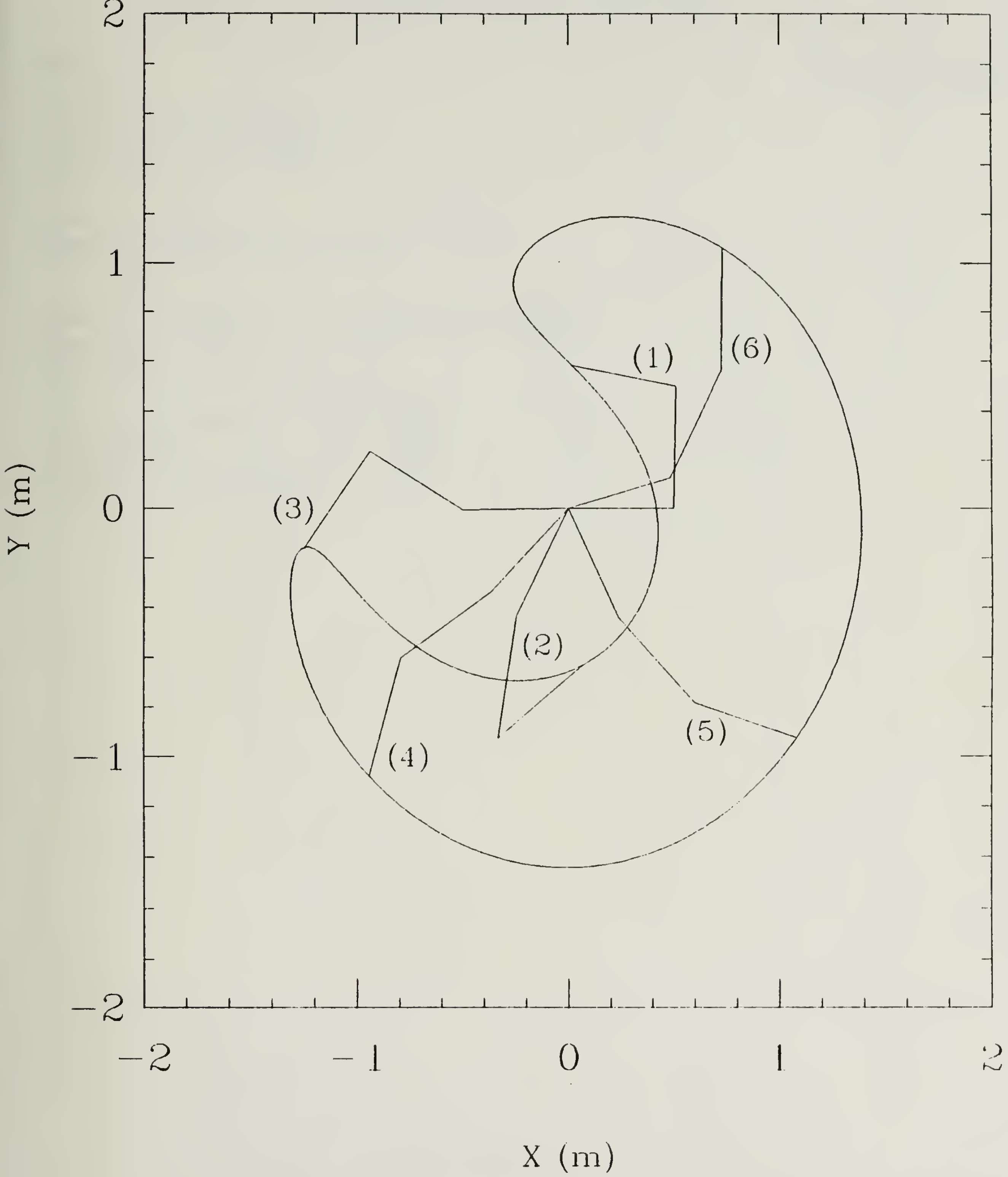


Figure 12. The closed path in the figure depicts an end-effector trajectory that will produce repeatable joint motion under pseudoinverse control. The figure also shows the configuration of the redundant manipulator at six different points along the trajectory.

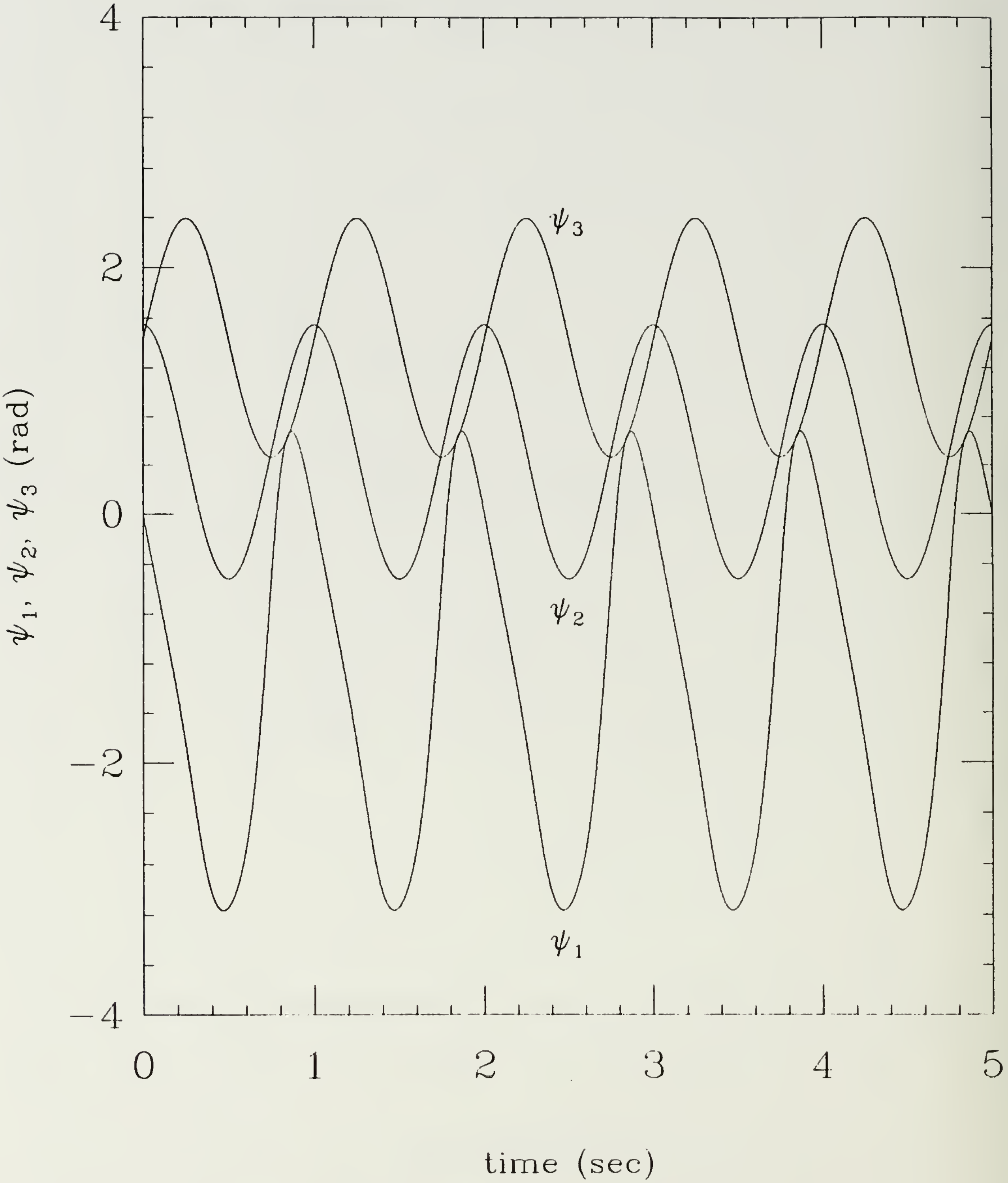


Figure 13. The repeatable joint trajectories of the three link redundant manipulator generated through pseudoinverse control of the end-effector trajectory in Figure 12.

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